LOCALLY DIVERGENT ORBITS OF MAXIMAL TORI AND VALUES OF FORMS AT INTEGRAL POINTS

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Abstract. Let $\mathbf{G}$ be a semisimple algebraic group defined over a number field K, $\mathbf{T}$ a maximal K-split torus of $\mathbf{G}$, $\mathcal{S}$ a finite set of valuations of K containing the archimedean ones, $\mathcal{O}$ the ring of $\mathcal{S}$-integers of K and $K_{\mathcal{S}}$ the direct product of the completions $K_v, v \in \mathcal{S}$. Denote $G = \mathbf{G}(K_{\mathcal{S}})$, $T = \mathbf{T}(K_{\mathcal{S}})$ and $\Gamma = \mathbf{G}(\mathcal{O})$. Let $T\pi(g)$ be a locally divergent orbit for the action of T on G/Γ by left translations. We prove: (1) if $\#S = 2$ then the closure $\overline{T\pi(g)}$ is a union of finitely many T-orbits all stratified in terms of parabolic subgroups of $\mathbf{G} \times \mathbf{G}$ and, therefore, $\overline{T\pi(g)}$ is homogeneous only if $T\pi(g)$ is closed; (2) if $\#\mathcal{S} > 2$ and K is not a CM-field then $\overline{T\pi(g)}$ is squeezed between closed orbits of two reductive groups of equal semisimple ranks implying that $\overline{T\pi(g)}$ is homogeneous when $\mathbf{G} = \mathbf{SL}_n$. As an application, if $f = (f_v)_{v\in\mathcal{S}} \in K_{\mathcal{S}}[x_1, \cdots, x_n]$, where f_v are non-pairwise proportional decomposable over K homogeneous forms, then $f(\mathcal{O}^n)$ is dense in $K_{\mathcal{S}}$.

1. Introduction

Let $\mathbf{G}$ be a semisimple algebraic group defined over a number field K. Let $\mathcal{S}$ be a finite set of (normalized) valuations of K containing all archimedean ones and $\mathcal{O}$ the ring of $\mathcal{S}$-integers of K. We denote by K_v, $v \in \mathcal{S}$, the completion of K with respect to v and by $K_{\mathcal{S}}$ the direct product of the topological fields K_v. Put $G = \mathbf{G}(K_{\mathcal{S}})$. The group G is naturally identified with the direct product of the locally compact groups $G_v = \mathbf{G}(K_v)$, $v \in \mathcal{S}$, and $\mathbf{G}(K)$ is diagonally imbedded in G. Let Γ be an $\mathcal{S}$-arithmetic subgroup of G, that is, $\Gamma \cap \mathbf{G}(\mathcal{O})$ have finite index in both Γ and $\mathbf{G}(\mathcal{O})$. The homogeneous space G/Γ is endowed with the quotient topology and has finite volume with respect to the Haar measure. Every closed subgroup H of G acts on G/Γ by left translations

$$h\pi(g) \overset{def}{=} \pi(hg),$$

where $\pi : G \to G/\Gamma$ is the quotient map. An orbit $H\pi(g)$ is called *divergent* if the orbit map $H \to G/\Gamma, h \mapsto h\pi(g)$, is proper, i.e., if $\{h_i\pi(g)\}$ leaves compacts of G/Γ whenever $\{h_i\}$ leaves compacts of H. It is clear that every divergent orbit is closed. We say that the closure $\overline{H\pi(g)}$ of $H\pi(g)$ in G/Γ is *homogeneous* if $\overline{H\pi(g)} = L\pi(g)$ for some closed subgroup L of G.

Fix a maximal K-split torus $\mathbf{T}$ of $\mathbf{G}$ and, for every $v \in \mathcal{S}$, a maximal K_v-split torus $\mathbf{T}_v$ of $\mathbf{G}$ containing $\mathbf{T}$. Recall that, given a field extension F/K, the

F-rank of $\mathbf{G}$, denoted by $\mathrm{rank}_F\mathbf{G}$, is the common dimension of the maximal F-split tori of $\mathbf{G}$. So, $\mathrm{rank}_{K_v}\mathbf{G} \geq \mathrm{rank}_K\mathbf{G}$ and $\mathrm{rank}_{K_v}\mathbf{G} = \mathrm{rank}_K\mathbf{G}$ if and only if $\mathbf{T} = \mathbf{T}_v$. Let $T_v = \mathbf{T}_v(K_v)$ and $T = \prod_{v\in\mathcal{S}} T_v \subset G$. An orbit $T\pi(g)$ is called *locally divergent* if $T_v\pi(g)$ is divergent for every $v \in \mathcal{S}$.

The locally divergent orbits, in general, and the closed locally divergent orbits, in particular, are classified by the following

Theorem 1.1. (cf.[T1, Theorem 1.4 and Corollary 1.5]) *With the above notation, we have:*

(a) *An orbit $T_v\pi(g)$ is divergent if and only if*

$$\mathrm{rank}_{K_v}\mathbf{G} = \mathrm{rank}_K\mathbf{G} \tag{1}$$

and

$$g \in \mathcal{Z}_G(T_v)\mathbf{G}(K), \tag{2}$$

where $\mathcal{Z}_G(T_v)$ is the centralizer of T_v in G. So, $T\pi(g)$ is locally divergent if and only if (1) and (2) hold for all $v \in \mathcal{S}$;

(b) *An orbit $T\pi(g)$ is locally divergent orbit and closed if and only if (1) holds for all $v \in \mathcal{S}$ and*

$$g \in \mathcal{N}_G(T)\mathbf{G}(K),$$

where $\mathcal{N}_G(T)$ is the normalizer of T in G.

The proof of Theorem 1.1 was preceded by [T-We], [We] and [T2] and by a result of G.A.Margulis for $\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$ (see [T-We, Appendix]).

Theorem 1.1 easily implies

Corollary 1.2. *A non-closed locally divergent T-orbit exists if and only if* $\mathrm{rank}_K\mathbf{G} > 0$*, $\#\mathcal{S} \geq 2$, and (1) is valid for all $v \in \mathcal{S}$.*

In the present paper we study the structure of the closures of the *non-closed* locally divergent T-orbits. It turns out that the cases $\#\mathcal{S} = 2$ and $\#\mathcal{S} > 2$ behave in drastically different ways. When $\#\mathcal{S} = 2$ the structure of $\overline{T\pi(g)}$ is similar to that of a toric variety: $T\pi(g)$ is open in $\overline{T\pi(g)}$ and $\overline{T\pi(g)}$ is a union of finitely many locally divergent T-orbits all stratified in terms of parabolic subgroups of $\mathbf{G} \times \mathbf{G}$ (Theorem 1.3). Therefore, $\overline{T\pi(g)}$ is homogeneous if and only if $T\pi(g)$ is closed (Corollary 1.6). On the other hand, when $\#\mathcal{S} > 2$ and $T\pi(g)$ is not closed $\overline{T\pi(g)}$ is never a finite union of T-orbits. In this case we prove (Theorem 1.8) that if K is not a CM-field [1] then $\overline{T\pi(g)}$ is squeezed between closed orbits of reductive subgroups of G of equal positive semisimple ranks. Hence if $\mathbf{G} = \mathbf{SL}_n$ then $\overline{T\pi(g)}$ is always homogeneous (Corollary 1.9).

During the recent years a number of problems from the Diophantine approximation of numbers have been reformulated in terms of action of maximal

[1] Recall that K is a CM-field if it is a quadratic extension K/F where the base field F is totally real but K is totally imaginary.

tori on G/Γ and, in some cases, successfully tackled. (See, for exemple, [M2], [E-K-L], [E-Kl], [Sha].) It is worth mentioning that the dynamics of the action of maximal split tori on G/Γ is closely related to the dynamics of the action of unipotent groups of G on G/Γ. Both actions have many complementary features but the latter is much better understood and motivates problems and conjectures about the former. For example, in contrast to the tori orbits, the unipotent orbits are always recurrent and, therefore, never divergent or locally divergent (cf. [M6] and [D]). As another example, if H is a subgroup generated by 1-parameter unipotent subgroups of G then $\overline{H\pi(g)}$ is homogenous. (See the proof of the Oppenheim Conjecture [M1], followed by [DM1], M.Ratner's results for arbitrary real Lie groups [Ra1] and [Ra2], and the corresponding results in $\mathcal{S}$-adic setting [BP], [MT], [Ra3] and [To4].) It was believed up to recently that $\overline{T\pi(g)}$ is homogenous whenever T is maximal (even higher dimensional) split torus and G/Γ does not admit rank 1 T-invariant factors. (See Margulis' [M3, Conjecture 1] for an exact formulation.) Sparse counter-examples to Margulis' conjecture have been given in [Mau] for $G = \mathrm{SL}_n(\mathbb{R}), n \geq 6$, and the action of multi-dimensional but non-maximal T, in [Sha] (see also [L-Sha, Theorem 1.5]) for $\mathrm{SL}_3(\mathbb{R})/\mathrm{SL}_3(\mathbb{Z})$ and maximal T, and in [T3] for direct products of $r \geq 2$ copies of $\mathrm{SL}_2(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{C})$ and the action of both maximal and non-maximal T. The main results from [T3] are extended in this paper to all semi-simple algebraic K-groups.

Let us give the exact formulations of our results starting with the case $\#\mathcal{S} = 2$. We will use some basic concepts from the theory of linear algebraic groups recalled in §2.3. So, let $\mathcal{S} = \{v_1, v_2\}$, $g = (g_{v_1}, g_{v_1}) \in G$ and $T\pi(g)$ be a locally divergent orbit. Let Π be a system of simple K-roots with respect to the maximal K-split torus $\mathbf{T}$. Given $\Psi \subset \Pi$, we denote by $\mathbf{P}_\Psi$ the corresponding to Ψ standard parabolic subgroup and by $\mathbf{P}_\Psi^-$ the opposite to $\mathbf{P}_\Psi$ parabolic subgroup. It is well known that $\mathbf{P}_\Psi$ (respectively, $\mathbf{P}_\Psi^-$) is a semidirect product of its unipotent radical $\mathbf{V}_\Psi$ (respectively, $\mathbf{V}_\Psi^-$) and the Levy subgroup $\mathcal{Z}_{\mathbf{G}}(\mathbf{T}_\Psi) = \mathbf{P}_\Psi \cap \mathbf{P}_\Psi^-$. Put

$$\mathcal{P}_\Psi(g) = \{\omega_1 \mathbf{P}_\Psi^- \omega_1^{-1} \times \omega_2 \mathbf{P}_\Psi \omega_2^{-1} | \omega_1, \omega_2 \in \mathcal{N}_{\mathbf{G}}(\mathbf{T}), g_{v_1} g_{v_2}^{-1} \in \omega_1 \mathbf{V}_\Psi^- \mathbf{P}_\Psi \omega_2^{-1}\}$$

and

$$\mathcal{P}(g) = \bigcup_{\Psi \subset \Pi} \mathcal{P}_\Psi(g).$$

The parabolic subgroups from the *finite* set $\mathcal{P}(g)$ are called *admissible with respect to* g. It is clear that $\mathcal{P}_\Pi(g) = \{\mathbf{G} \times \mathbf{G}\}$ and $\mathcal{P}_\emptyset(g)$ consists of minimal parabolic K-subgroups of $\mathbf{G} \times \mathbf{G}$.

To every $\mathbf{P} \in \mathcal{P}(g)$ we associate a locally divergent T-orbit as follows. If $\mathbf{P} = \omega_1 \mathbf{P}_\Psi^- \omega_1^{-1} \times \omega_2 \mathbf{P}_\Psi \omega_2^{-1}$ and $g_{v_1} g_{v_2}^{-1} = \omega_1 v_\Psi^- z_\Psi v_\Psi \omega_2^{-1}$, where $v_\Psi^- \in \mathbf{V}_\Psi^-$, $z_\Psi \in \mathcal{Z}_{\mathbf{G}}(\mathbf{T}_\Psi)$ and $v_\Psi \in \mathbf{V}_\Psi$, we put

$$\mathrm{Orb}_g(\mathbf{P}) \overset{def}{=} T(\omega_1 (v_\Psi^-)^{-1} \omega_1^{-1}, \omega_2 v_\Psi \omega_2^{-1}) \pi(g). \tag{3}$$

It is easy to see that $\mathrm{Orb}_g(\mathbf{P})$ is a well-defined locally divergent orbit (Lemma 5.1). Note that $\mathrm{Orb}_g(\mathbf{G}\times\mathbf{G}) = T\pi(g)$.

Theorem 1.3. *With the above notation, let $\mathbf{P}\in\mathcal{P}(g)$. Then*

$$\overline{\mathrm{Orb}_g(\mathbf{P})} = \bigcup_{\mathbf{P}'\in\mathcal{P}(g),\ \mathbf{P}'\subset\mathbf{P}} \mathrm{Orb}_g(\mathbf{P}').$$

In particular,

$$\overline{T\pi(g)} = \bigcup_{\mathbf{P}\in\mathcal{P}(g)} \mathrm{Orb}_g(\mathbf{P}).$$

Theorem 1.3 implies easily:

Corollary 1.4. *Let $\mathbf{P}\in\mathcal{P}(g)$. Choose a $\widetilde{\mathbf{P}}\in\mathcal{P}(g)$ with $\mathrm{Orb}_g(\widetilde{\mathbf{P}}) = \mathrm{Orb}_g(\mathbf{P})$ and being minimal with this property. Then*

$$\overline{\mathrm{Orb}_g(\mathbf{P})}\setminus\mathrm{Orb}_g(\mathbf{P}) = \bigcup_{\mathbf{P}'\in\mathcal{P}(g),\ \mathbf{P}'\subsetneqq\widetilde{\mathbf{P}}} \overline{\mathrm{Orb}_g(\mathbf{P}')}.$$

In particular, $T\pi(g)$ is open in its closure.

The closed T-orbits in $\overline{T\pi(g)}$ are parameterized by the minimal parabolic subgroups of $\mathbf{G}\times\mathbf{G}$ belonging to $\mathcal{P}(g)$. Namely, we have:

Corollary 1.5. *If $\mathbf{P}$ is minimal in $\mathcal{P}(g)$ then $\mathrm{Orb}_g(\mathbf{P})$ is closed and $\mathbf{P}$ is a minimal parabolic subgroup of $\mathbf{G}\times\mathbf{G}$. Moreover, $\mathcal{P}_\emptyset(g)\neq\emptyset$ and $\{\mathrm{Orb}_g(\mathbf{P}) : \mathbf{P}\in\mathcal{P}_\emptyset(g)\}$ is the set of all closed T-orbits in $\overline{T\pi(g)}$.*

We get the following refinement of Theorem 1.1 (b):

Corollary 1.6. *The following conditions are equivalent:*

(a) *$T\pi(g)$ is closed,*
(b) *$\overline{T\pi(g)}$ is homogenous,*
(c) *$g\in\mathcal{N}_G(T)\mathbf{G}(K)$,*
(d) *$T\pi(g) = \mathrm{Orb}_g(\mathbf{P})$ for some minimal $\mathbf{P}\in\mathcal{P}(g)$,*
(e) *$T\pi(g) = \mathrm{Orb}_g(\mathbf{P})$ for every $\mathbf{P}\in\mathcal{P}(g)$.*

The map $\mathbf{P}\mapsto\mathrm{Orb}_g(\mathbf{P})$, $\mathbf{P}\in\mathcal{P}(g)$, is not injective, in general, but it becomes injective if $g_{v_1}g_{v_2}^{-1}$ belongs to a non-empty Zariski dense subset of $\mathbf{G}$.

Corollary 1.7. *For every $\Psi\subset\Pi$, denote by n_Ψ the number of parabolic subgroups containing $\mathbf{T}$ and conjugated to $\mathbf{P}_\Psi$. We have*

(a) *The number of different T-orbits in $\overline{T\pi(g)}$ is bounded from above by $\sum_{\Psi\subset\Pi} n_\Psi^2$ and the number of different closed T-orbits in $\overline{T\pi(g)}$ is bounded from above by $n_\emptyset^2$;*

(b) *There exists a non-empty Zariski dense subset* $\Omega \subset \mathbf{G}(K)$ *such that if* $g_{v_1}g_{v_2}^{-1} \in \Omega$ *then the map* $\mathrm{Orb}_g(\cdot)$ *is injective and* $\overline{T\pi(g)}$ *is a union of exactly* $\sum_{\Psi \subset \Pi} n_\Psi^2$ *pairwise different* T*-orbits and among them exactly* $n_\emptyset^2$ *are closed.*

Recall that the semi-simple K-rank of a reductive K-group $\mathbf{H}$, denoted by $\mathrm{s.s.rank}_K(\mathbf{H})$, is equal to $\mathrm{rank}_K\mathfrak{D}(\mathbf{H})$ where $\mathfrak{D}(\mathbf{H})$ is the derived subgroup of $\mathbf{H}$.

The main result for $\#\mathcal{S} > 2$ is the following

Theorem 1.8. *Let* $\#\mathcal{S} > 2$, K *be not a* CM*-field and* $T\pi(g)$ *be a locally divergent non-closed orbit. Then there exist* h_1 *and* $h_2 \in \mathcal{N}_G(T)\mathbf{G}(K)$ *and reductive* K*-subgroups* $\mathbf{H}_1$ *and* $\mathbf{H}_2$ *of* $\mathbf{G}$ *such that*

$$\mathrm{s.s.rank}_K(\mathbf{H}_1) = \mathrm{s.s.rank}_K(\mathbf{H}_2) > 0, \tag{4}$$

and

$$h_2H_2\pi(e) \subset \overline{T\pi(g)} \subset h_1H_1\pi(e),$$

where $H_1 = \mathbf{H}_1(K_\mathcal{S})$, $H_2 = \mathbf{H}_2(K_\mathcal{S})$ *and the orbits* $h_1H_1\pi(e)$ *and* $h_2H_2\pi(e)$ *are closed and* T*-invariant.*

The above theorem implies

Corollary 1.9. *With the assumptions of Theorem 1.8, if* $\mathbf{G} = \mathbf{SL}_n$ *then* $\overline{T\pi(g)}$ *is homogeneous. Moreover, if* $g = (g_{v_1}, \cdots, g_{v_r}) \in G$, *where* $\mathcal{S} = \{v_1, \cdots, v_r\}$, $T\pi(g)$ *is dense in* G/Γ *if and only if* $\bigcap_{i=1}^{r-1} \mathcal{Z}_\mathbf{T}(\omega_i g_{v_i}(g_{v_r})^{-1})$ *is finite for all choices of* $\omega_i \in \mathcal{N}_\mathbf{G}(\mathbf{T})$.

In §8.2 we prove Theorem 8.1 which shows on the example of $\mathbf{G} = \mathbf{SL}_2$ that the claims of Theorem 1.8 and Corollary 1.9 are not valid for CM fields. Moreover, using the restriction of scalars functor, we get that $\overline{T\pi(g)}$ as in the formulation of Theorem 1.8 is, in general, not homogeneous. Theorem 8.1 provides counter-examples to Margulis' conjecture which differ from the counter-examples in [Mau], [Sha] and [L-Sha] in the following sense: in [Mau], [Sha] and [L-Sha] $\overline{T\pi(g)} \setminus T\pi(g)$ is contained in a union of 2 closed orbits of proper subgroups of G while in Theorem 8.1 $\overline{T\pi(g)} \setminus T\pi(g)$ is not contained in a countable union of closed orbits of proper subgroups of G.

As an application, we describe the closures of the values at $\mathcal{S}$-integral points of decomposable over K homogeneous forms. First, we introduce the necessary notation and formulate a general conjecture. Let $K_\mathcal{S}[\vec{x}\,]$ be the ring of polynomials in $n \geq 2$ variables $\vec{x} = (x_1, \dots, x_n)$ with coefficients from the topological ring $K_\mathcal{S}$. Note that $K_\mathcal{S}[\vec{x}] = \prod_{v \in \mathcal{S}} K_v[\vec{x}]$. The ring $K[\vec{x}]$ is identified with its diagonal imbedding in $K_\mathcal{S}[\vec{x}]$. Let $f(\vec{x}) = (f_v(\vec{x}))_{v \in \mathcal{S}} \in K_\mathcal{S}[\vec{x}]$. We suppose

that every $f_v(\vec{x}) = l_1^{(v)}(\vec{x}) \cdots l_m^{(v)}(\vec{x})$, where $l_1^{(v)}(\vec{x}), \ldots, l_m^{(v)}(\vec{x})$ are linearly independent over K_v linear forms in $K_v[\vec{x}]$. (So, $m \leq n$.) It is easy to see that if $f(\vec{x}) = c \cdot h(\vec{x})$, where $h(\vec{x}) \in K[\vec{x}]$ and $c \in K_{\mathcal{S}}$, then $f(\mathcal{O}^n)$ is discrete in $K_{\mathcal{S}}$. In fact, the opposite is also true: the discreteness of $f(\mathcal{O}^n)$ in $K_{\mathcal{S}}$ implies that $f_v(\vec{x}), v \in \mathcal{S}$, are all proportional to a polynomial $h(\vec{x}) \in K[\vec{x}]$ ([T1, Theorem 1.8]). It is natural to ask what is the closure of $f(\mathcal{O}^n)$ in $K_{\mathcal{S}}$ if $f_v(\vec{x}), v \in \mathcal{S}$, are not all proportional to some polynomial with coefficients from K.

Conjecture. With f as above, let $\#\mathcal{S} > 2$ and K be not a CM-field. Suppose that $f_v(\vec{x}), v \in \mathcal{S}$, are not all proportional to a non-zero polynomial with coefficients from K. Then $f(\mathcal{O}^n)$ is dense in $K_{\mathcal{S}}$.

We say that f_v is decomposable over K if the linear forms $l_1^{(v)}(\vec{x}), \ldots, l_m^{(v)}(\vec{x})$ are all with coefficients from K. Using Corollary 1.9, we prove

Theorem 1.10. *Conjecture 1 is true if every f_v, $v \in \mathcal{S}$, is decomposable over K.*

It is shown in §8.3 that the analog of Theorem 1.10 (and, therefore, that of the above conjecture) is not always true if $\#\mathcal{S} = 2$ or $\#\mathcal{S} \geq 2$ and K is a CM-field. If $\overline{f(\mathcal{O}^n)} = K_{\mathcal{S}}$ it is a natural problem to understand the distribution of $f(\mathcal{O}^n)$ in $K_{\mathcal{S}}$. Presumably, it is a matter of a uniform distribution.

2. Preliminaries: notation and some basic concepts

2.1. **Numbers.** As usual, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the non-negative integer, integer, rational, real and complex numbers, respectively. Also, $\mathbb{N}_+ = \{x \in \mathbb{N} : x > 0\}$ and $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$.

In this paper K is a number field, that is, a finite extension of $\mathbb{Q}$. All valuations of K which we consider are supposed to be *normalized* (see [CF, ch.2, §7]) and, therefore, pairwise non-equivalent. If v is a valuation of K then K_v is the completion of K with respect to v and $|\cdot|_v$ is the corresponding norm on K_v. Recall that if $K_v = \mathbb{R}$ (respectively, $K_v = \mathbb{C}$) then $|\cdot|_v$ is the absolute value on $\mathbb{R}$ (respectively, the square of the absolute value on $\mathbb{C}$). If v is non-archimedean then $\mathcal{O}_v = \{x \in K_v : |\ x\ |_v \leq 1\}$ is the ring of integers of K_v.

We fix a finite set $\mathcal{S} = \{v_1, \cdots, v_r\}$ of valuations of K containing all archimedean valuations of K. The archimedean valuations in $\mathcal{S}$ will be denoted by $\mathcal{S}_\infty$. We also denote $\mathcal{S}_f = \mathcal{S} \setminus \mathcal{S}_\infty$.

Sometimes we will write K_i instead of K_{v_i} and $|\cdot|_i$ instead of $|\cdot|_{v_i}$.

We denote by $\mathcal{O}$ the ring of $\mathcal{S}$-integers of K, i.e., $\mathcal{O} = K \bigcap (\bigcap_{v \notin \mathcal{S}} \mathcal{O}_v)$. Also, $\mathcal{O}_\infty = K \bigcap (\bigcap_{v \notin \mathcal{S}_\infty} \mathcal{O}_v)$ is the ring of integers of K.

Let $K_{\mathcal{S}} \overset{def}{=} \prod_{v \in \mathcal{S}} K_v$. The field K is a dense subfield of the topological ring $K_{\mathcal{S}}$ and $\mathcal{O}$ is a lattice in $K_{\mathcal{S}}$.

As usual, if R is a ring R^* denotes the multiplicative group of units of R.

2.2. **Groups.** Further on, we use boldface letters to denote the algebraic groups defined over K (shortly, the K-algebraic groups or the algebraic K-groups). Let $\mathbf{H}$ be a K-algebraic group. As usual, $\mathcal{R}_u(\mathbf{H})$ (respectively, $\mathrm{Lie}(\mathbf{G})$) stands for the unipotent radical (respectively, the Lie algebra) of $\mathbf{H}$. Given $v \in \mathcal{S}$, we write $H_v \stackrel{def}{=} \mathbf{H}(K_v)$ or simply H_i if $\mathcal{S} = \{v_1, \cdots, v_r\}$ and $v = v_i$. Put $H \stackrel{def}{=} \mathbf{H}(K_{\mathcal{S}})$. On every H_v we have *Zariski topology* induced by the Zariski topology on $\mathbf{H}$ and *Hausdorff topology* induced by the Hausdorff topology on K_v. The formal product of the Zariski (resp., the Hausdorff) topologies on H_v, $v \in S$, is the Zariski (respectively, the Hausdorff) topology on H. In order to distinguish the two topologies, all topological notions connected with the first one will be used with the prefix "Zariski".

The algebraic groups in this paper are supposed to be linear. Every K-algebraic group $\mathbf{H}$ is a Zariski closed K-subgroup of $\mathbf{SL}_l$ for some $l \in \mathbb{N}_+$. The group $\mathbf{SL}_l$ itself is identified with $\mathrm{SL}_l(\Omega)$ where Ω is a universal domain, i.e. Ω is an algebraically closed field of infinite transcendental degree over $\mathbb{Q}$ containing K and all K_v. We denote by $\mathbf{GL}_1$ the 1-dimensional $\mathbb{Q}$-split torus and by $\mathbf{D}_l$ the subgroup of diagonal matrices in $\mathbf{SL}_l$. So, $\mathbf{D}_l$ is isomorphic over $\mathbb{Q}$ to $\mathbf{GL}_1^{l-1}$. Moreover, $\mathbf{H}(\mathcal{O}) = \mathbf{SL}_l(\mathcal{O}) \cap \mathbf{H}$. A subgroup Δ of H is called $\mathcal{S}$-*arithmetic* if Δ and $\mathbf{H}(\mathcal{O})$ are commensurable, that is, if $\Delta \cap \mathbf{H}(\mathcal{O})$ has finite index in both Δ and $\mathbf{H}(\mathcal{O})$. Recall that if $\mathbf{H}$ is semisimple then Δ is a lattice in H, i.e. H/Δ has finite Haar measure.

The Zariski connected component of the identity $e \in \mathbf{H}$ is denoted by $\mathbf{H}^\bullet$. In the case of a real Lie group L the connected component of the identity is denoted by L°.

If A and B are subgroups of an abstract group C then $\mathcal{N}_A(B)$ (resp., $\mathcal{Z}_A(B)$) is the normalizer (resp., the centralizer) of B in A.

2.3. K**-roots.** In this paper $\mathbf{G}$ is a connected, semisimple, K-isotropic algebraic group and $\mathbf{T}$ is a maximal K-split torus in $\mathbf{G}$. The imbedding of $\mathbf{G}$ in $\mathbf{SL}_l$ (see §2.2) is chosen in such a way that $\mathbf{T} = \mathbf{G} \cap \mathbf{D}_l$ and $\mathbf{T}(\mathcal{O}) = \mathbf{G} \cap \mathbf{D}_l(\mathcal{O}) \cong (\mathcal{O}^*)^{\mathrm{rank}_K \mathbf{G}}$.

We denote by $\Phi (\equiv \Phi(\mathbf{T}, \mathbf{G}))$ the system of K-roots with respect to $\mathbf{T}$. Let Φ^+ be a system of positive K-roots in Φ and Π be the set of simple roots in Φ^+. (We refer to [B, §21.1] for the standard definitions related to the K-roots.)

If $\chi \in \Phi$ we let $\mathfrak{g}_\chi$ be the corresponding root-space in $\mathrm{Lie}(\mathbf{G})$. For every $\alpha \in \Pi$ we define a projection $\pi_\alpha : \Phi \to \mathbb{Z}$ by $\pi_\alpha(\chi) = n_\alpha$ where $\chi = \sum_{\beta \in \Pi} n_\beta \beta$.

Let $\Psi \subset \Pi$ and $\mathbf{T}_\Psi \stackrel{def}{=} (\bigcap_{\alpha \in \Psi} \ker(\alpha))^\bullet$. We denote by $\mathbf{P}_\Psi$ the (standard) parabolic subgroup corresponding to Ψ and by $\mathbf{P}_\Psi^-$ the opposite parabolic subgroup corresponding to Ψ. The centralizer $\mathcal{Z}_{\mathbf{G}}(\mathbf{T}_\Psi)$ is a common Levi subgroup of $\mathbf{P}_\Psi$ and $\mathbf{P}_\Psi^-$, $\mathbf{P}_\Psi = \mathcal{Z}_{\mathbf{G}}(\mathbf{T}_\Psi) \ltimes \mathcal{R}_u(\mathbf{P}_\Psi)$ and $\mathbf{P}_\Psi^- = \mathcal{Z}_{\mathbf{G}}(\mathbf{T}_\Psi) \ltimes \mathcal{R}_u(\mathbf{P}_\Psi^-)$. We will often use the simpler notation $\mathbf{V}_\Psi \stackrel{def}{=} \mathcal{R}_u(\mathbf{P}_\Psi)$ and $\mathbf{V}_\Psi^- \stackrel{def}{=} \mathcal{R}_u(\mathbf{P}_\Psi^-)$. Recall

that

$$\mathrm{Lie}(\mathbf{V}_{\boldsymbol{\Psi}}) = \bigoplus_{\exists\alpha\in\Pi\setminus\Psi,\ \pi_\alpha(\chi)>0} \mathfrak{g}_\chi, \tag{5}$$

$$\mathrm{Lie}(\mathbf{V}^-_{\boldsymbol{\Psi}}) = \bigoplus_{\exists\alpha\in\Pi\setminus\Psi,\ \pi_\alpha(\chi)<0} \mathfrak{g}_\chi, \tag{6}$$

and

$$\mathrm{Lie}(\mathcal{Z}_{\mathbf{G}}(\mathbf{T}_\Psi)) = \mathrm{Lie}(\mathcal{Z}_{\mathbf{G}}(\mathbf{T})) \oplus \bigoplus_{\forall\alpha\in\Pi\setminus\Psi,\ \pi_\alpha(\chi)=0} \mathfrak{g}_\chi. \tag{7}$$

It is well known that the map $\Psi \mapsto \mathbf{P}_\Psi$ is a bijection between the subsets of Π and the parabolic subgroups of $\mathbf{G}$ containing $\mathbf{B}$, cf. [B, §21.11]. Note that $\mathbf{P}_\emptyset$, $\mathbf{P}_\emptyset{}^-$ are minimal parabolic subgroups and $\mathbf{G} = \mathbf{P}_\Pi = \mathbf{P}^-_\Pi$.

Given $\alpha \in \Phi$ we let (α) be the set of roots which are positive multiple of α. Then $\mathfrak{g}_{(\alpha)} \stackrel{def}{=} \bigoplus_{\beta\in(\alpha)} \mathfrak{g}_\beta$ is the Lie algebra of a unipotent group denoted by $\mathbf{U}_{(\alpha)}$. Given $\Psi \subset \Pi$, let Ψ' be the set of all non-divisible positive roots χ such that $\exists\alpha \in \Delta \setminus \Psi,\ \pi_\alpha(\chi) > 0$. Then the product morphism (in any order) $\prod\limits_{\chi\in\Psi'} \mathbf{U}_{(\chi)} \to \mathbf{V}_\Psi$ is an isomorphism of K-varieties [B, 21.9].

It follows from the above definitions that $\Psi_1 \subset \Psi_2 \ \Leftrightarrow\ \mathbf{P}_{\Psi_1} \subset \mathbf{P}_{\Psi_2} \ \Leftrightarrow\ \mathbf{V}_{\Psi_1} \supset \mathbf{V}_{\Psi_2} \ \Leftrightarrow\ \mathcal{Z}_{\mathbf{G}}(\mathbf{T}_{\Psi_1}) \subset \mathcal{Z}_{\mathbf{G}}(\mathbf{T}_{\Psi_2})$. Let $\mathbf{V}_{[\Psi_2\setminus\Psi_1]} \stackrel{def}{=} \mathcal{Z}_{\mathbf{G}}(\mathbf{T}_{\Psi_2}) \cap \mathbf{V}_{\Psi_1}$ and $\mathbf{V}^-_{[\Psi_2\setminus\Psi_1]} \stackrel{def}{=} \mathcal{Z}_{\mathbf{G}}(\mathbf{T}_{\Psi_2}) \cap \mathbf{V}^-_{\Psi_1}$. It is easy to see that

$$\mathbf{V}_{\Psi_1} = \mathbf{V}_{\Psi_2}\mathbf{V}_{[\Psi_2\setminus\Psi_1]} = \mathbf{V}_{[\Psi_2\setminus\Psi_1]}\mathbf{V}_{\Psi_2}. \tag{8}$$

Recall that the Weyl group $\mathcal{W} \stackrel{def}{=} \mathcal{N}_{\mathbf{G}}(\mathbf{T})/\mathcal{Z}_{\mathbf{G}}(\mathbf{T})$ acts by conjugation simply transitively on the set of all minimal parabolic K-subgroups of $\mathbf{G}$ containing $\mathbf{T}$. When this does not lead to confusion, we will identify the elements from $\mathcal{W}$ with their representatives from $\mathcal{N}_{\mathbf{G}}(\mathbf{T})$. It is easy to see that $\mathcal{W}_\Psi = \mathcal{N}_{\mathcal{Z}_{\mathbf{G}}(\mathbf{T}_\Psi)}(\mathbf{T})/\mathcal{Z}_{\mathbf{G}}(\mathbf{T})$ is the Weyl group of $\mathcal{Z}_{\mathbf{G}}(\mathbf{T}_\Psi)$. Note that $\mathcal{W} = \mathcal{W}_\emptyset$.

We will denote by ω_0 the element from $\mathcal{W}$ such that $\omega_0\mathbf{P}_\emptyset\omega_0^{-1} = \mathbf{P}^-_\emptyset$.

3. On the group of units of $\mathcal{O}$

Recall that $\mathcal{S} = \{v_1, \cdots, v_r\}$, $r \geq 2$, $K_i = K_{v_i}$ and $K_{\mathcal{S}} = \prod\limits_i K_i$. By the $\mathcal{S}$-adic version of Dirichlet's unit theorem, the $\mathbb{Z}$-rank of $\mathcal{O}^*$ is equal to $r-1$. Moreover, if $K^1_{\mathcal{S}} = \{(x_1, \cdots, x_r) \in K^*_{\mathcal{S}} : |x_1|_1 \cdots |x_r|_r = 1\}$ then $\mathcal{O}^*$ is a lattice of $K^1_{\mathcal{S}}$.

For every $m \in \mathbb{N}_+$, we denote $\mathcal{O}^*_m = \{\xi^m | \xi \in \mathcal{O}^*\}$. The next proposition follows easily from the compactness of $K^1_{\mathcal{S}}/\mathcal{O}^*_m$.

Proposition 3.1. *For every $m \in \mathbb{N}_+$ there exists a constant $\kappa_m > 1$ such that given $(a_i) \in K^1_{\mathcal{S}}$ there exists $\xi \in \mathcal{O}^*_m$ satisfying*

$$\frac{1}{\kappa_m} \leq |\xi a_i|_i \leq \kappa_m$$

for all $1 \leq i \leq r$.

Let $\mathcal{S}_\infty = \{v_1, \cdots, v_{r'}\}$ and $\mathcal{S}_f = \{v_{r'+1}, \cdots, v_r\}$. So, $K_1 = \mathbb{R}$ or $\mathbb{C}$. In the next proposition $p : K^1_{\mathcal{S}} \to K^*_1$ is the natural projection and $L \overset{def}{=} \overline{p(\mathcal{O}^*)}$.

Proposition 3.2. *With the above notation, we have:*

(1) *If $r = 2$ then $L^\circ = \{1\}$;*
(2) *Let $r \geq 3$. We have:*
   (a) *$L^\circ \neq \{1\}$. In particular, $L^\circ = \mathbb{R}_+$ if $K_1 = \mathbb{R}$;*
   (b) *Let $K_1 = \mathbb{C}$.*
      (i) *If $L^\circ = \mathbb{R}_+$ then K is a* CM*-field;*
      (ii) *Suppose that K is not a* CM*-field and $L \neq \mathbb{C}^*$. Then L° coincides with the unit circle group in $\mathbb{C}$ unless $r = 3$.*

Proof. (1) follows easily from the compactness of $K^1_{\mathcal{S}}/\mathcal{O}^*_m$.

(2) If $r \geq 3$ in view of Dirichlet's unit theorem $\mathcal{O}^*$ contains a subgroup of $\mathbb{Z}$-rank 2. Therefore $p(\mathcal{O}^*)$ is not discrete, proving that $L^\circ \neq \{1\}$.

Let $K_1 = \mathbb{C}$. Suppose that $L^\circ = \mathbb{R}_+$. Therefore L is a finite extension of L°. Hence there exists m such that $p(\mathcal{O}^*_m)$ is a dense subgroup of L°. Let F be the number field generated over $\mathbb{Q}$ by $\mathcal{O}^*_m$. Then F is proper subfield of K and its unit group has the same $\mathbb{Z}$-rank as that of K, i.e. K has a "unit defect". It is known that the fields with "unit defect" are exactly the CM-fields (cf.[Re]).

It remains to consider the case when K is not a CM-field, $L \neq \mathbb{C}^*$ and $r > 3$. Since L° is a 1- dimensional subgroup of $\mathbb{C}^*$ we need to prove that L° couldn't be a spiral. This will be deduced from the following six exponentials theorem due to Siegel: if x_1, x_2, x_3 are three complex numbers linearly independent over $\mathbb{Q}$ and y_1, y_2 are two complex numbers linearly independent over $\mathbb{Q}$ then at least one of the six numbers $\{e^{x_i y_j} : 1 \leq i \leq 3, 1 \leq j \leq 2\}$ is transcendental.

Now, suppose by the contrary that L° is a spiral, that is, $L^\circ = \{e^{t(a+\mathrm{i}b)} : t \in \mathbb{R}\}$ for some a and $b \in \mathbb{R}^*$. Since $r > 3$ there exist ξ_1, ξ_2 and $\xi_3 \in p(\mathcal{O}^*)$ which are multiplicatively independent over the integers. We may suppose that $\xi_1 = e^{a+\mathrm{i}}$, $\xi_2 = e^{u(a+\mathrm{i}b)}$ and $\xi_3 = e^{v(a+\mathrm{i}b)}$ where u and $v \in \mathbb{R}^*$ and $\mathrm{i} = \sqrt{-1}$. Remark that $\{1, u, v\}$ are linearly independent over $\mathbb{Q}$, $\{a + \mathrm{i}b, \mathrm{i}b\}$ are linearly independent over $\mathbb{Q}$, and the six numbers ξ_1, ξ_2, ξ_3, $\frac{\xi_1}{|\xi_1|}$, $\frac{\xi_2}{|\xi_2|}$, $\frac{\xi_3}{|\xi_3|}$ are all algebraic. This contradicts the six exponentials theorem. $\square$

If $K_1 = \mathbb{C}$ and K is not a CM-field, it is not difficult to give examples when L° is the circle group and when $L = \mathbb{C}^*$.

Examples.1) For every $n \geq 1$, let $f_n(x) = (x^2 - (\sqrt{n^2+1} + n)x + 1)(x^2 - (-\sqrt{n^2+1} + n)x + 1)$. Then $f_n(x)$ is an irreducible polynomial in $\mathbb{Q}[X]$ with

two real and two (conjugated) complex roots. Let $K_1 = \mathbb{Q}(\alpha_n)$ where α_n is one of the complex roots of $f_n(x)$. Then L° is the circle group of $\mathbb{C}^*$.

2) It is easy to see that if K is a totally imaginary, Galois, non-CM-number field of degree ≥ 6 then $L = \mathbb{C}^*$.

Finally, the following is quite plausible:

Conjecture 2. L° is never a spiral.

In response to a question of the author, Federico Pellarin observed that Conjecture 2 follows from the still open four exponentials conjecture. The use of the six exponentials theorem in our proof of Proposition 3.2 is inspired by Pellarin's argument.

4. Accumulations points for locally divergent orbits

As in the introduction Γ is an $\mathcal{S}$-arithmetic subgroup of G, $T = \mathbf{T}(K_{\mathcal{S}})$ and T acts on G/Γ by left translations. In the next lemma $\mathbf{T}(\mathcal{O})$ is identified with $(\mathcal{O}^*)^{\mathrm{rank}_K \mathbf{G}}$ via the isomorphism from §2.3.

Lemma 4.1. *Let $h \in \mathbf{G}(K)$. The following assertions hold:*

(a) *There exists a positive integer m such that $\xi\pi(h) = \pi(h)$ for all $\xi \in (\mathcal{O}_m^*)^{\mathrm{rank}_K \mathbf{G}}$;*
(b) *If h_i is a sequence in G such that $\{\pi(h_i)\}$ converges to an element from G/Γ then the sequence $\{\pi(h_i h)\}$ admits a converging to an element from G/Γ subsequence.*

The lemma is an easy consequence from the commensurability of Γ and $h\Gamma h^{-1}$.

4.1. Main proposition.

We need the following general

Proposition 4.2. *Let $n \in \mathcal{N}_{\mathbf{G}}(\mathbf{T})$ and $\Psi \subset \Pi$. The following conditions are equivalent:*

(i) *$n \in w_0 \mathcal{W}_\Psi$;*
(ii) *$\mathbf{V}_\emptyset^- w_0 n \mathbf{P}_\Psi$ is Zariski dense in $\mathbf{G}$;*
(iii) *$w_0 n \mathbf{V}_\Psi (w_0 n)^{-1} \subset \mathbf{V}_\emptyset$.*

Proof. The implications (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) are easy.

Let (ii) holds. Then $n^{-1}\mathbf{V}_\emptyset n \mathbf{P}_\Psi$ is Zariski dense. Since $n^{-1}\mathbf{V}_\emptyset n$ and $\mathbf{P}_\Psi$ are $\mathbf{T}$-invariant

$$\mathrm{Lie}(n^{-1}\mathbf{V}_\emptyset n) = \mathrm{Lie}(\mathbf{V}_\Psi^-) + \mathrm{Lie}(n^{-1}\mathbf{V}_\emptyset n \cap \mathbf{P}_\Psi). \tag{9}$$

Therefore

$$n^{-1}\mathbf{V}_\emptyset n = \mathbf{V}_\Psi^- (n^{-1}\mathbf{V}_\emptyset n \cap \mathbf{P}_\Psi). \tag{10}$$

Since $n^{-1}\mathbf{V}_\emptyset n$ is a product of root groups, if $n^{-1}\mathbf{V}_\emptyset n \cap \mathbf{V}_\Psi \neq \{e\}$ then, in view of (10), $n^{-1}\mathbf{V}_\emptyset n$ contains two opposite root groups which is not possible. Therefore

$$n^{-1}\mathbf{V}_\emptyset n \cap \mathbf{V}_\Psi = \{e\}.$$

This implies that $n^{-1}\mathbf{V}_\emptyset n \cap \mathbf{P}_\Psi = n^{-1}\mathbf{V}_\emptyset n \cap \mathscr{Z}_{\mathbf{G}}(\mathbf{T}_\Psi)$. In view of (9) $n^{-1}\mathbf{V}_\emptyset n \cap \mathscr{Z}_{\mathbf{G}}(\mathbf{T}_\Psi)$ is a maximal unipotent subgroup of $\mathscr{Z}_{\mathbf{G}}(\mathbf{T}_\Psi)$. Let $n' \in \mathcal{W}_\Psi$ be such that $n'(n^{-1}\mathbf{V}_\emptyset n \cap \mathscr{Z}_{\mathbf{G}}(\mathbf{T}_\Psi))n'^{-1} \subset \mathbf{V}_\emptyset^-$. Since n' normalizes $\mathbf{V}_\Psi^-$, it follows from (10) that

$$n'n^{-1}\mathbf{V}_\emptyset n n'^{-1} = \mathbf{V}_\emptyset^-$$

which imlies (i).

Suppose that (iii) holds. Then $(w_0 n)^{-1}\mathbf{P}_\Psi w_0 n \supset \mathbf{V}_\Psi$. Hence, $w_0 n \in \mathbf{P}_\Psi$ (cf. [B, 14.22]). Therefore $w_0 n \in \mathcal{W}_\Psi$, proving (i). $\square$

Further on, $g = (g_1, g_2, \cdots, g_r) \in G$ where $g_i \in G_i$. If $g_i \in \mathbf{G}(K)$ and $h = (h_1, \cdots, h_r) \in G$, writing $\pi(hg_i)$ we mean that g_i is identified with its the diagonal imbedding in G, so that, $hg_i = (h_1 g_i, \cdots, h_r g_i)$.

Our main proposition is the following:

Proposition 4.3. *Let $\#\mathcal{S} \geq 2$, $\mathrm{rank}_K \mathbf{G} = \mathrm{rank}_{K_1}\mathbf{G} = \mathrm{rank}_{K_2}\mathbf{G}$, g_1 and $g_2 \in \mathbf{G}(K)$ and Ψ be a proper subset of Π. Let $(s_n, t_n, e, \cdots, e) \in T$ be a sequence and $C > 1$ be a constant such that for all n we have: $|\alpha(s_n)|_1 > \frac{1}{C}$ for all $\alpha \in \Pi$, $|\alpha(t_n)|_2 \underset{n}{\to} 0$ for all $\alpha \in \Pi \setminus \Psi$ and $\frac{1}{C} < |\alpha(t_n)|_2 < C$ for all $\alpha \in \Psi$. Then the sequence $(s_n, t_n, e, \cdots, e)\pi(g)$ is bounded in G/Γ if and only if the following conditions are satisfied:*

(i) *$g_1 g_2^{-1} \in \mathbf{V}_\Psi^- \mathbf{P}_\Psi$, and*
(ii) *there exists a constant $C' > 1$ such that $\frac{1}{C'} < |\alpha(s_n)|_1 \cdot |\alpha(t_n)|_2 < C'$ for all $\alpha \in \Delta$ and all n.*

Proof. $\Leftarrow$) Suppose that (i) and (ii) hold. Then $g_1 = vpg_2$ where $v \in \mathbf{V}_\Psi^-(K)$ and $p \in \mathbf{P}_\Psi(K)$. It follows from (ii), Lemma 4.1 and Proposition 3.1 that there exists a sequence $d_n \in \mathrm{Stab}_T\{\pi(pg_2)\}$ such that the sequence $(s_n d_n^{-1}, t_n d_n^{-1}, d_n^{-1}, \cdots, d_n^{-1})$ is bounded in T.

Now

$$\begin{aligned}
&(s_n, t_n, e, \cdots, e) \cdot (g_1, g_2, \cdots, g_r)\pi(e) = \\
&(s_n, t_n, e, \cdots, e) \cdot (vpg_2, p^{-1}pg_2, g_3 \cdots, g_r)\pi(e) = \\
&(s_n v s_n^{-1}, t_n p^{-1} t_n^{-1}, e, \cdots, e) \cdot (s_n, t_n, e, \cdots, e) \cdot \\
&(e, e, g_3(pg_2)^{-1}, \cdots, g_n(pg_2)^{-1})\pi(pg_2) = \\
&(s_n v s_n^{-1}, t_n p^{-1} t_n^{-1}, e, \cdots, e) \cdot (s_n d_n^{-1}, t_n d_n^{-1}, e, \cdots, e) \cdot \\
&(e, e, g_3(pg_2)^{-1} d_n^{-1}, \cdots, g_n(pg_2)^{-1} d_n^{-1})\pi(pg_2).
\end{aligned}$$

Note that $t_np^{-1}t_n^{-1}$ is bounded in G_2. By (ii) $|\alpha(s_n)|_1 \underset{n}{\to} \infty$ for all $\alpha \in \Pi \setminus \Psi$. Therefore $s_nvs_n^{-1} \to e$ in G_1. Now using the choice of d_n we conclude that $(s_n, t_n, e, \cdots, e)\pi(g)$ is bounded in G/Γ.

$\Rightarrow$) Let $(s_n, t_n, e, \cdots, e)\pi(g)$ be bounded. By Bruhat decomposition $g_1g_2^{-1} = v^-w_0np$ where $v^- \in \mathbf{V}_\emptyset^-$, $n \in \mathcal{N}_{\mathbf{G}}(\mathbf{T})$, $p \in \mathbf{P}_\Psi$ and $w_0\mathbf{V}_\emptyset^-w_0^{-1} = \mathbf{V}$. Suppose that $\mathbf{V}_\emptyset^-w_0n\mathbf{P}_\Psi$ is not Zariski dense in $\mathbf{G}$. In view of Proposition 4.2 there exists a root χ such that $\mathbf{U}_\chi \subset \mathbf{V}_\Psi$ and $\chi \circ \mathrm{Int}(w_0n)^{-1}$ is a negative root. Let $d_n \in T \cap \mathrm{Stab}_G\{pg_2\}$ be such that $\{t_nd_n^{-1}\}$ is bounded in G_2 and $\{d_n\}$ is bounded in every G_i, $i \geq 3$. Then $|\chi((w_0n)^{-1}s_nw_0n)|_1$ is bounded from above and $|\chi(d_n^{-1})|_1 \to 0$. Hence

$$|\chi((w_0n)^{-1}s_nw_0n)\chi(d_n^{-1})|_1 \to 0. \tag{11}$$

But

$$\begin{aligned}
&(s_n, t_n, e, \cdots, e)\pi(g) = \\
&(s_n, t_n, e, \cdots, e) \cdot (v^-w_0n, p^{-1}, g_3(pg_2)^{-1}, \cdots, g_r(pg_2)^{-1})\pi(pg_2) = \\
&((s_nv^-s_n^{-1}w_0n)((w_0n)^{-1}s_n(w_0n)d_n^{-1}), (t_np^{-1}t_n^{-1})t_nd_n^{-1}, g_3(pg_2)^{-1}d_n^{-1}, \\
&\cdots, g_r(pg_2)^{-1}d_n^{-1})\pi(pg_2).
\end{aligned}$$

It follows from (11) and from the choice of $\{d_n\}$ that the sequence

$$\{(s_nv^-s_n^{-1}w_0n)((w_0n)^{-1}s_n(w_0n)d_n^{-1})\}$$

is unbounded in G_1 and the sequences $\{(t_np^{-1}t_n^{-1})t_nd_n^{-1}\}$, $\{g_3(pg_2)^{-1}d_n^{-1}\}$, $\cdots$, $\{g_r(pg_2)^{-1}d_n^{-1}\}$ are bounded in $G_2, G_3, \cdots, G_r$, respectively. Since $T_1\pi(pg_2)$ is divergent, we get that $(s_n, t_n, e, \cdots, e)\pi(g)$ is unbounded which is a contradiction. Therefore $\mathbf{V}_\emptyset^-w_0n\mathbf{P}_\Psi$ is Zariski dense in $\mathbf{G}$. So, in view of Proposition 4.2, $\mathbf{V}_\emptyset^-w_0n\mathbf{P}_\Psi = \mathbf{V}_\Psi^-\mathbf{P}_\Psi = \mathbf{V}_\emptyset^-\mathbf{P}_\Psi$, proving (i).

Let $g_1 = v^-pg_2$, where $v^- \in \mathbf{V}_\Psi^-$, $p \in \mathbf{P}_\Psi$. Then

$$\begin{aligned}
&(s_n, t_n, e, \cdots, e)\pi(g) = ((s_nv^-s_n^{-1})(s_nd_n^{-1}), (t_np^{-1}t_n^{-1})(t_nd_n^{-1}), \\
&g_3(pg_2)^{-1}d_n^{-1}, \cdots, g_r(pg_2)^{-1}d_n^{-1})\pi(pg_2).
\end{aligned}$$

Using that $s_nv^-s_n^{-1}$ is bounded in G_1, $t_np^{-1}t_n^{-1}$ and $t_nd_n^{-1}$ are both bounded in G_2 and the projections of d_n into G_i, $i \geq 3$, are all bounded, it follows from the assumptions that $(s_n, t_n, e, \cdots, e)\pi(g)$ is bounded in G/Γ and $T_1\pi(g)$ is divergent that $s_nd_n^{-1}$ is bounded in G_1. Hence there exists $C_1 > 1$ such that $\frac{1}{C_1} < |\alpha(s_nd_n^{-1})|_1 \cdot |\alpha(t_nd_n^{-1})|_2 < C_1$ for all $\alpha \in \Pi$. By Artin's product formula $\prod_{v \in \mathcal{V}} |\alpha(d_n)|_v = 1$ where $\mathcal{V}$ is the set of all normalized valuations of K. This implies (ii). $\square$

The above proposition implies:

Corollary 4.4. *Let $s_n \in T_1$ and $t_n \in T_2$ be such that for every $\alpha \in \Pi$ each of the sequences $|\alpha(s_n)|_1$ and $|\alpha(t_n)|_2$ converges to an element from $\mathbb{R} \cup \infty$. We suppose that g_1 and $g_2 \in \mathbf{G}(K)$ and that $(s_n, t_n, e, \cdots, e)\pi(g)$ converges in G/Γ. Then there exist $\Psi \subset \Pi$ and $\omega_1, \omega_2 \in \mathcal{W}$ with the following properties:*

(i) $\omega_1^{-1}\mathbf{P}_\Psi^-\omega_1(K_1)\times\omega_2^{-1}\mathbf{P}_\Psi\omega_2(K_2)=$
$=\{(x,y)\in G_1\times G_2 : \mathrm{Int}(s_n,t_n)(x,y)$ *is bounded in* $G_1\times G_2\}$,

(ii) $g_1g_2^{-1}\in\omega_1^{-1}\mathbf{V}_\Psi^-\mathbf{P}_\Psi\omega_2$.

(iii) *If* $g_1g_2^{-1}=\omega_1^{-1}v_\Psi^- z_\Psi v_\Psi\omega_2$, *where* $v_\Psi^-\in\mathbf{V}_\Psi^-(K)$, $z_\Psi\in\mathcal{Z}_{\mathbf{G}}(\mathbf{T}_\Psi)(K)$ *and* $v_\Psi\in\mathbf{V}_\Psi(K)$, *then*

$$\lim_n(s_n,t_n,e,\cdots,e)\pi(g)=(d_1\omega_1^{-1}(v_\Psi^-)^{-1}\omega_1,d_2\omega_2^{-1}v_\Psi\omega_2,e,\cdots,e)\pi(g),$$

where $d_1\in T_1$ *and* $d_2\in T_2$.

Proof. There exists a parabolic K-subgroup $\mathbf{P}$ containing $\mathbf{T}$ such that $\mathbf{P}(K_1)=\{x\in\mathbf{G}(K_1):\mathrm{Int}(s_n)x$ is bounded$\}$. Let $\Psi\subset\Pi$ and $\omega_1\in\mathcal{W}$ be such that $\omega_1^{-1}\mathbf{P}_1\omega_1=\mathbf{P}_\Psi^-$. Similarly, we find $\Psi'\subset\Pi$ and $\omega_2\in\mathcal{W}$ such that $\omega_2\mathbf{P}_{\Psi'}\omega_2^{-1}(K_2)=\{x\in\mathbf{G}(K_2):\mathrm{Int}(t_n)x$ is bounded$\}$.

Put $\widetilde{g}=(\omega_1^{-1}g_1,\omega_2^{-1}g_2,g_3,\cdots,g_r)$, $\widetilde{s}_n=\omega_1^{-1}s_n\omega_1$ and $\widetilde{t}_n=\omega_2^{-1}t_n\omega_2$. Then $(\widetilde{s}_n,\widetilde{t}_n,e,\cdots,e)\pi(\widetilde{g})$ converges, $\mathbf{P}_\Psi^-(K_1)=\{x\in\mathbf{G}(K_1):\mathrm{Int}(\widetilde{s}_n)x$ is bounded$\}$ and $\mathbf{P}_{\Psi'}(K_2)=\{x\in\mathbf{G}(K_2):\mathrm{Int}(\widetilde{t}_n)x$ is bounded$\}$. So, there exists a constant $C>1$ such that $C>|\alpha(\widetilde{s}_n)|_1>\frac{1}{C}$ for all $\alpha\in\Psi$, $|\alpha(\widetilde{s}_n)|_1\to\infty$ for all $\Pi\setminus\Psi$, $|\alpha(\widetilde{t}_n)|_2\to 0$ for all $\Pi\setminus\Psi'$ and $C>|\alpha(\widetilde{t}_n)|_2>\frac{1}{C}$ for all $\alpha\in\Psi'$. Passing to a subsequence and replacing if necessary C by a larger constant we may suppose that for every $\alpha\in\Phi$ either $C>|\alpha(\widetilde{t}_n)|_2>\frac{1}{C}$ for all n or $|\alpha(\widetilde{t}_n)|_2$ is converging to 0 or ∞. It follows from Proposition 4.3(ii) that $\frac{1}{C}<|\alpha(\widetilde{s}_n)|_1\cdot|\alpha(\widetilde{t}_n)|_2<C$ for all $\alpha\in\Pi$ and n. This implies easily that $\Psi=\Psi'$. In view of Proposition 4.3(i) $g_1g_2^{-1}\in\omega_1\mathbf{V}_\Psi^-\mathbf{P}_\Psi\omega_2^{-1}$. Hence (i) and (ii) hold.

Let $\omega_1^{-1}g_1=v_\Psi^- z_\Psi v_\Psi\omega_2^{-1}g_2$ as in the formulation of the corollary. Then writing

$$(\widetilde{s}_n,\widetilde{t}_n,e,\cdots,e)\pi(\widetilde{g})=$$
$$(\widetilde{s}_n,\widetilde{t}_n,e,\cdots,e)(v_\Psi^- z_\Psi,v_\Psi^-,\cdots,g_r(v_\Psi\omega_2^{-1}g_2)^{-1})\pi(v_\Psi\omega_2^{-1}g_2),$$

a similar calculation as in the proof of Proposition 4.3 shows that

$$\lim(\widetilde{s}_n,\widetilde{t}_n,e,\cdots,e)\pi(\widetilde{g})=\pi(d_1'z_\Psi v_\Psi\omega_2^{-1}g_2,d_2'v_\Psi\omega_2^{-1}g_2,g_3\cdots,g_r)$$

where $(d_1',d_2')\in T_1\times T_2$. This implies (iii). □

5. Locally divergent orbits for $\#\mathcal{S}=2$

In this section $g=(g_1,g_2)$ and $T\pi(g)$ is a locally divergent orbit. We use the notation $\mathcal{P}(g)$, $\mathcal{P}_\Psi(g)$ and $\mathrm{Orb}_g(\mathbf{P})$ as defined in the Introduction [2].

Lemma 5.1. *Let* $\mathbf{P}=\omega_1\mathbf{P}_\Psi^-\omega_1^{-1}\times\omega_2\mathbf{P}_\Psi\omega_2^{-1}\in\mathcal{P}_\Psi(g)$ *and* $g_1g_2^{-1}=\omega_1v_\Psi^- z_\Psi v_\Psi\omega_2^{-1}$, *where* $v_\Psi^-\in\mathbf{V}_\Psi^-$, $z_\Psi\in\mathcal{Z}_{\mathbf{G}}(\mathbf{T}_\Psi)$, $v_\Psi\in\mathbf{V}_\Psi$, *and* ω_1 *and* $\omega_2\in\mathcal{N}_{\mathbf{G}}(\mathbf{T})$. *We have:*

[2] Remark that $\mathbf{P}_\Psi^-$ and $\mathbf{P}_\Psi$ are not always conjugated and , therefore, in the definition of $\mathcal{P}_\Psi(g)$ we can not replace $\mathbf{P}_\Psi^-$ by $\mathbf{P}_\Psi$. Indeed, let $\mathbf{G}$ be a simple K-split algebraic group of type $D_l, l\geq 4$, and $\alpha\in\Pi$ be such that $\omega_0(\alpha)\neq-\alpha$. Then choosing $\Psi=\{\alpha\}$ it is easy to see that $\mathbf{P}_\Psi^-$ is not conjugated to $\mathbf{P}_\Psi$.

(a) $(\omega_1(v_\Psi^-)^{-1}\omega_1^{-1}g_1, \omega_2 v_\Psi \omega_2^{-1} g_2) \in \bigcap_{v\in\mathcal{S}} \mathcal{Z}_G(T_v)\mathbf{G}(K)$, *that is, the orbit (3) is well-defined and locally divergent;*

(b) *If* $w \in \mathcal{W}\times\mathcal{W}$ *then* $w\mathbf{P}w^{-1} \in \mathcal{P}(wg)$ *and*

$$w\mathrm{Orb}_g(\mathbf{P}) = \mathrm{Orb}_{wg}(w\mathbf{P}w^{-1}). \tag{12}$$

Proof. The assertion (a) of the lemma is invariant under multiplication of (g_1, g_2) from the left by elements from $\mathcal{Z}_G(T)$. In view of Theorem 1.1(a), this reduce the proof to the case when g_1 and $g_2 \in \mathbf{G}(K)$. Since $\mathcal{N}_{\mathbf{G}}(\mathbf{T}) = \mathcal{N}_{\mathbf{G}}(\mathbf{T})(K)\mathcal{Z}_{\mathbf{G}}(\mathbf{T})$ we have $\omega_i = \widetilde{\omega}_i a_i$ where $\widetilde{\omega}_i \in \mathcal{N}_{\mathbf{G}}(\mathbf{T})(K)$ and $a_i \in \mathcal{Z}_{\mathbf{G}}(\mathbf{T})$. So,

$$\widetilde{\omega}_1^{-1} g_1 g_2^{-1} \widetilde{\omega}_2 = a_1 v_\Psi^- z_\Psi v_\Psi a_2^{-1} = \widetilde{v}_\Psi^- \widetilde{z}_\Psi \widetilde{v}_\Psi \in \mathbf{G}(K),$$

where $\widetilde{v}_\Psi^- \in \mathbf{V}_\Psi^-$, $\widetilde{z}_\Psi \in \mathcal{Z}_{\mathbf{G}}(\mathbf{T}_\Psi)$ and $\widetilde{v}_\Psi \in \mathbf{V}_\Psi$. Since the product map $\mathbf{V}_\Psi^-(K)\times \mathcal{Z}_{\mathbf{G}}(\mathbf{T}_\Psi)(K)\times\mathbf{V}_\Psi(K) \to (\mathbf{V}_\Psi^-\mathcal{Z}_{\mathbf{G}}(\mathbf{T}_\Psi)\mathbf{V}_\Psi)(K)$ is bijective, we get that $\widetilde{v}_\Psi^-, \widetilde{z}_\Psi$, and $\widetilde{v}_\Psi$ are K-rational. It remains to note that $\omega_1(v_\Psi^-)^{-1}\omega_1^{-1} = \widetilde{\omega}_1(\widetilde{v}_\Psi^-)^{-1}\widetilde{\omega}_1^{-1}$ and $\omega_2 v_\Psi \omega_2^{-1} = \widetilde{\omega}_2 \widetilde{v}_\Psi \widetilde{\omega}_2^{-1}$.

The part (b) follows from the definition of the orbit $\mathrm{Orb}_g(\mathbf{P})$ by a simple computation. $\square$

5.1. **Proof of Theorem 1.3.** In view of (12), it is enough to prove the theorem for $\mathbf{P} = \mathbf{P}_\Psi^- \times \mathbf{P}_\Psi$. In this case $g_1 g_2^{-1} = v_\Psi^- z_\Psi v_\Psi$, where $v_\Psi^- \in \mathbf{V}_\Psi^-$, $z_\Psi \in \mathcal{Z}_{\mathbf{G}}(\mathbf{T}_\Psi)$ and $v_\Psi \in \mathbf{V}_\Psi$, and

$$\mathrm{Orb}_g(\mathbf{P}) = T(z_\Psi, e)\pi(v_\Psi g_2).$$

The orbit $\mathcal{Z}_G(T_\Psi)\pi(v_\Psi g_2)$ is closed, it contains $\mathrm{Orb}_g(\mathbf{P})$, and $\mathcal{Z}_{\mathbf{G}}(\mathbf{T}_\Psi)$ is a reductive K-algebraic group . Since $\mathcal{Z}_G(T_\Psi)\pi(v_\Psi g_2)$ is homeomorphic to $\mathcal{Z}_G(T_\Psi)/\Delta$, where Δ is an $\mathcal{S}$-arithmetic subgroup of $\mathcal{Z}_G(T_\Psi)$, the T-orbits on $\mathcal{Z}_G(T_\Psi)\pi(v_\Psi g_2)$ contained in $\mathrm{Orb}_g(\mathbf{P})$ are described by Corollary 4.4 applied to $\mathcal{Z}_{\mathbf{G}}(\mathbf{T}_\Psi)$. Let Tm be such an orbit. There exists $\Psi' \subset \Psi$ such that up to a conjugation of z_Ψ by an element from $\mathcal{W}_\Psi$ we have $z_\Psi = v_{[\Psi\setminus\Psi']}^- z_{\Psi'} v_{[\Psi\setminus\Psi']}$, where $v_{[\Psi\setminus\Psi']}^- \in \mathbf{V}_{[\Psi\setminus\Psi']}^-$, $z_{\Psi'} \in \mathcal{Z}_{\mathbf{G}}(\mathbf{T}_{\Psi'})$ and $v_{[\Psi\setminus\Psi']} \in \mathbf{V}_{[\Psi\setminus\Psi']}$ (see (8)), and

$$Tm = T(z_{\Psi'}, e)\pi(v_{[\Psi\setminus\Psi']} v_\Psi g_2).$$

It is clear that $\mathbf{P}_{\Psi'}^- \times \mathbf{P}_{\Psi'} \in \mathcal{P}(g)$ and

$$Tm = \mathrm{Orb}_g(\mathbf{P}_{\Psi'}^- \times \mathbf{P}_{\Psi'}),$$

proving the theorem. $\square$

5.1.1. *Proof of Corollaries 1.5 and 1.6.* By (12) and Theorem 1.1(a) we may (and will) suppose that g_1 and $g_2 \in \mathbf{G}(K)$ and $\mathbf{P} = \mathbf{P}_\Psi^- \times \mathbf{P}_\Psi$. If $g_1 g_2^{-1} = v_\Psi^- z_\Psi v_\Psi$ where $v_\Psi^- \in \mathbf{V}_\Psi^-$, $z_\Psi \in \mathcal{Z}_{\mathbf{G}}(\mathbf{T}_\Psi)$ and $v_\Psi \in \mathbf{V}_\Psi$ then

$$\mathrm{Orb}_g(\mathbf{P}) = T(z_\Psi, e)\pi(v_\Psi g_2). \tag{13}$$

If $\mathbf{P}$ is minimal among the subgroups in $\mathcal{P}(g)$ then $\mathrm{Orb}_g(\mathbf{P})$ is closed in view of Theorem 1.3. It follows from Theorem 1.1 (b) that $z_\Psi \in \mathcal{W}_\Psi$. So,

$$g_1 g_2^{-1} = v_\Psi^-(z_\Psi v_\Psi z_\Psi^{-1}) z_\Psi \in \mathbf{P}_\emptyset^- \mathbf{P}_\emptyset z_\Psi$$

and $\mathbf{P}_{\emptyset}^{-} \times z_{\Psi}^{-1}\mathbf{P}_{\emptyset}z_{\Psi} \in \mathcal{P}(g)$. Since $\mathbf{P}$ is minimal in $\mathcal{P}(g)$ and $\mathbf{P}_{\emptyset}^{-} \times z_{\Psi}^{-1}\mathbf{P}_{\emptyset}z_{\Psi} \subset \mathbf{P}_{\Psi}^{-} \times \mathbf{P}_{\Psi}$ we get that $\mathbf{P}$ is a minimal parabolic subgroup of $\mathbf{G} \times \mathbf{G}$. This complete the proof of Corollary 1.5.

Concerning the proof of Corollary 1.6, it is easy to see that $(a) \Leftrightarrow (b) \Leftrightarrow (e)$ in view of Theorem 1.3, $(a) \Leftrightarrow (c)$ in view of Theorem 1.1 and $(a) \Leftrightarrow (d)$ in view of Corollary 1.5. $\square$

5.1.2. *Proof of Corollary 1.7.* The part (a) of the corollary is a direct consequence from Theorem 1.3.

Let us prove (b). Denote by $\mathcal{P}$ the set of all parabolic subgroups

$$\omega_1\mathbf{P}_{\Psi}^{-}\omega_1^{-1} \times \omega_2\mathbf{P}_{\Psi}\omega_2^{-1}$$

where ω_1 and $\omega_1 \in \mathcal{N}_{\mathbf{G}}(\mathbf{T})$ and $\Psi \subset \Pi$. Let $\Omega_1 = \bigcap_{(\omega_1,\omega_1)\in\mathcal{W}\times\mathcal{W}} \omega_1^{-1}\mathbf{P}_{\emptyset}^{-}\mathbf{P}_{\emptyset}\omega_2$. Then Ω_1 is W-invariant, Zariski open, non-empty and $\mathcal{P} = \mathcal{P}(g)$ if and only if $g_1g_2^{-1} \in \Omega_1$.

Let $\mathbf{P}$ and $\mathbf{P}' \in \mathcal{P}(g)$ and $g_1g_2^{-1} \in \Omega_1$. Suppose that the set of minimal parabolic subgroups of $\mathbf{P}$ containing $\mathbf{T}$ coincides with the set of minimal parabolic subgroups of $\mathbf{P}'$ containing $\mathbf{T}$. We may suppose that $\mathbf{P} = \mathbf{P}_{\Psi}^{-} \times \mathbf{P}_{\Psi}$. Since $\mathbf{P}_{\emptyset}^{-} \times \mathbf{P}_{\emptyset} \subset \mathbf{P}'$ we get that $\mathbf{P}' = \mathbf{P}_{\Psi'}^{-} \times \mathbf{P}_{\Psi'}$ for some $\Psi' \subset \Pi$. The group $\mathcal{W}_{\Psi}$ (respectively, $\mathcal{W}_{\Psi'}$) acts simply transitively on the minimal parabolic subgroups of $\mathbf{P}_{\Psi}$ (respectively, $\mathbf{P}_{\Psi'}$) containing $\mathbf{T}$. Therefore $\mathcal{W}_{\Psi} = \mathcal{W}_{\Psi'}$. But $\mathbf{P}_{\Psi} = \mathbf{P}_{\emptyset}\mathcal{W}_{\Psi}\mathbf{P}_{\emptyset}$ and $\mathbf{P}_{\Psi'} = \mathbf{P}_{\emptyset}\mathcal{W}_{\Psi'}\mathbf{P}_{\emptyset}$. Hence $\mathbf{P}_{\Psi} = \mathbf{P}_{\Psi'}$ and $\mathbf{P} = \mathbf{P}'$, i.e. each $\mathbf{P} \in \mathcal{P}(g)$ is uniquely determined by its minimal parabolic subgroups. Therefore the map $\mathrm{Orb}_g(\cdot)$ is injective if and only if its restriction to the set of minimal parabolic subgroups is injective.

Let $\Delta = g_2\Gamma g_2^{-1}$. It is well known that the product map $\mathbf{V}_{\emptyset}^{-} \times \mathcal{Z}_{\mathbf{G}}(\mathbf{T}) \times \mathbf{V}_{\emptyset} \to \mathbf{G}$ is a K-rational isomorphism. Let $p : \mathbf{G} \to \mathbf{V}_{\emptyset}^{-}$ be the natural projection. Choose a non-archimedean completion F of K different from K_1 and K_2. Since p is K-rational the closure $\overline{p(\Delta)}$ of $p(\Delta)$ in $\mathbf{V}_{\emptyset}^{-}(F)$ (for the Hausdorff topology on $\mathbf{V}_{\emptyset}^{-}(F)$ induced by the topology on F) is compact. Therefore there exists a non-empty, $\mathcal{W}$-invariant, open (for the Hausdorff topology on $\mathbf{G}(F)$) subset $\Omega_2 \subset \mathbf{G}(F)$ with the following properties: if $x \in \Omega_2$, $\omega \in \mathcal{W} \setminus \{e\}$ and $\omega x \omega^{-1} = v^{-}zv$, where $v^{-} \in \mathbf{V}_{\emptyset}^{-}(F)$, $z \in \mathcal{Z}_{\mathbf{G}}(\mathbf{T})(F)$ and $v \in \mathbf{V}_{\emptyset}(F)$, then $p(\omega^{-1}v^{-1}\omega) \notin \overline{p(\Delta)}$.

Let $\Omega = \Omega_1 \cap \Omega_2$. It is clear that Ω is non-empty and Zariski dense in $\mathbf{G}$. Let $g_1 \in \Omega g_2$. Let $\mathbf{P}$ and $\mathbf{P}' \in \mathcal{P}_{\emptyset}$ be such that $\mathrm{Orb}_g(\mathbf{P}) = \mathrm{Orb}_g(\mathbf{P}')$. It remains to prove that $\mathbf{P} = \mathbf{P}'$. In view of (12), we may assume that $\mathbf{P} = \mathbf{P}_{\emptyset}^{-} \times \mathbf{P}_{\emptyset}$ and $\mathbf{P}' = \omega_1^{-1}\mathbf{P}_{\emptyset}^{-}\omega_1 \times \omega_2^{-1}\mathbf{P}_{\emptyset}\omega_2$, where ω_1 and $\omega_1 \in \mathcal{W}$. Then

$$g_1g_2^{-1} = v^{-}zv = \omega_1^{-1}v_1^{-}z_1v_1\omega_2,$$

where v^{-} and $v_1^{-} \in \mathbf{V}_{\emptyset}^{-}$, z and $z_1 \in \mathcal{Z}_{\mathbf{G}}(\mathbf{T})$, and v and $v_1 \in \mathbf{V}_{\emptyset}$. Using (3) we get $(t_1, t_2) \in T$ and $\delta \in \Delta$ such that

$$t_1zv = \omega_1^{-1}z_1v_1\omega_2\delta \text{ and } t_2v = \omega_1^{-1}v_1\omega_2\delta. \tag{14}$$

This implies
$$\omega_1^{-1}\omega_2 = (t_1 z t_2^{-1})(\omega_2^{-1} z_1 \omega_2) \in \mathscr{Z}_{\mathbf{G}}(\mathbf{T}).$$
Therefore $\omega_1 = \omega_2 = \omega$. Using (14)
$$(\omega^{-1} v_1^{-1}\omega) t_2 v \in \Delta.$$
Finally, in view of the choice of Ω_2, we get $\omega = e$, i.e. $\mathbf{P} = \mathbf{P}'$. $\square$

6. Locally divergent orbits for $\#\mathcal{S} > 2$

Further on we suppose that $\mathrm{rank}_K \mathbf{G} = \mathrm{rank}_{K_v}\mathbf{G}$ for all $v \in \mathcal{S}$.

6.1. **Horospherical subgroups.** Let $t \in T_v, v \in \mathcal{S}$. We set
$$W^+(t) = \{x \in G_v : \lim_{n\to+\infty} t^{-n} x t^n = e\},$$

$$W^-(t) = \{x \in G_v : \lim_{n\to+\infty} t^{n} x t^{-n} = e\}$$
and
$$Z(t) = \{x \in G_v : t^n x t^{-n},\ n \in \mathbb{Z},\ \text{is bounded}\}.$$
Then $W^+(t)$ (respectively, $W^-(t)$) is the positive (respectively, negative) horospherical subgroup of G_v corresponding to t.

The following proposition is well known and easy to prove.

Proposition 6.1. *With the above notation, there exist opposite parabolic K-subgroups $\mathbf{P}$ and $\mathbf{P}^-$ containing $\mathbf{T}$ such that $W^+(t) = \mathcal{R}_u(\mathbf{P})(K_v)$, $W^-(t) = \mathcal{R}_u(\mathbf{P}^-)(K_v)$ and $Z(t) = (\mathbf{P}\cap\mathbf{P}^-)(K_v)$.*

Lemma 6.2. *Let $\Psi \subset \Pi$, $\sigma \in \mathbf{G}(K)$ and $1 \le s_1 < s_2 \le r$ where $\#\mathcal{S} = r$. There exists a sequence $t_n \in \mathbf{T}(K)\cap\sigma\Gamma\sigma^{-1}$ with the following properties:*

(a) *if $\alpha \in \Psi$ then $\alpha(t_n) = 1$ for all n;*
(b) *if $\alpha \in \Pi \setminus \Psi$ then $\lim_n |\alpha(t_n^{-1})|_i = 0$ when $1 \le i \le s_1$, $\lim_n |\alpha(t_n)|_i = 0$ when $s_1 + 1 \le i \le s_2$, and $|\alpha(t_n)|_i$ is bounded when $s_2 + 1 \le i \le r$.*

Proof. The lemma follows from Proposition 3.1 and the commensurability of $\mathbf{T}(\mathcal{O})$ and $\mathbf{T}(K)\cap\sigma\Gamma\sigma^{-1}$. $\square$

Proposition 6.3. *Let Ψ, σ, s_1, and s_2 be as in the formulation of Lemma 6.2. Also let $u_i \in \mathbf{V}_\Psi^-(K_i)$ if $1 \le i \le s_1$, $u_i \in \mathbf{V}_\Psi(K_i)$ if $s_1 + 1 \le i \le s_2$ and $g_i \in G_i$ if $s_2 + 1 \le i \le r$. Then the closure of the orbit $T\pi(u_1\sigma, \cdots, u_{s_2}\sigma, g_{s_2+1}, \cdots, g_r)$ contains $\pi(\sigma, \cdots, \sigma, g_{s_2+1}, \cdots, g_r)$.*

Proof. Let $t_n \in \mathbf{T}(K)\cap\sigma\Gamma\sigma^{-1}$ be as in Lemma 6.2. Passing to a subsequence we suppose that the projection of the sequence t_n in T_i is convergent for every $i > s_2$. Since $t_n\pi(\sigma) = \pi(\sigma)$ and in view of (5), we get
$$\lim_n t_n \pi(u_1\sigma, \cdots, u_{s_2}\sigma, g_{s_2+1}, \cdots, g_r) = (e, \cdots, e, h_{s_2+1}, \cdots, h_r)\pi(\sigma),$$

where $h_i = \lim_n t_n g_i \sigma^{-1} t_n^{-1}, i > s_2$. Using once again the convergence of t_n in every T_i, $i > s_2$, we get

$$\lim_n t_n^{-1}(e, \cdots, e, h_{s_2+1}, \cdots, h_r)\pi(\sigma) = \pi(\sigma, \cdots, \sigma, g_{s_2+1}, \cdots, g_r).$$

□

Lemma 6.4. *Let $\Psi \subset \Pi$ and $g \in \mathbf{G}(K)$.*

(a) *We have $g = \omega z v_+ v_-$, where $\omega \in \mathcal{N}_{\mathbf{G}}(\mathbf{T})(K)$, $z \in \mathcal{Z}_{\mathbf{G}}(\mathbf{T}_\Psi)(K)$, $v_+ \in \mathbf{V}_\Psi^+(K)$ and $v_- \in \mathbf{V}_\Psi^-(K)$. Moreover,*

$$\mathcal{Z}_{\mathbf{T}_\Psi}(g) = \mathcal{Z}_{\mathbf{T}_\Psi}(v_-) \cap \mathcal{Z}_{\mathbf{T}_\Psi}(v_+) \cap \mathcal{Z}_{\mathbf{T}_\Psi}(\omega).$$

(b) *With $g = \omega z v_+ v_-$ as in (a), suppose that $\dim \mathcal{Z}_{\mathbf{T}_\Psi}(g) \geq \dim \mathcal{Z}_{\mathbf{T}_\Psi}(\theta g)$ for every $\theta \in \mathcal{N}_{\mathbf{G}}(\mathbf{T})$. Then*

$$\mathcal{Z}_{\mathbf{T}_\Psi}(\omega)^\bullet \supset \mathcal{Z}_{\mathbf{T}_\Psi}(g)^\bullet = \big(\mathcal{Z}_{\mathbf{T}_\Psi}(v_-) \cap \mathcal{Z}_{\mathbf{T}_\Psi}(v_+)\big)^\bullet.$$

Proof. The part (a) follow freely from the existence and the uniqueness of the Bruhat decomposition for reductive K-algebraic groups (cf. [B, Theorem 21.15]). The part (b) follows immediately from (a). □

Lemma 6.5. *Let $g \in \mathbf{G}(K)$ be such that $\dim \mathcal{Z}_{\mathbf{T}}(g) \geq \dim \mathcal{Z}_{\mathbf{T}}(\theta g)$ for all $\theta \in \mathcal{N}_{\mathbf{G}}(\mathbf{T})$. Let Λ be a subset of $\Phi(\mathbf{T}, \mathbf{G})$ and $\mathbf{S} = \big(\bigcap_{\alpha\in\Lambda} \ker \alpha\big)^\bullet$. There exist systems of simple roots Π and Π' in Φ and subsets $\Psi \subset \Pi$ and $\Psi' \subset \Pi'$ with the following properties:*

(a) $\mathbf{S} = \mathbf{T}_\Psi = \mathbf{T}_{\Psi'}$*;*

(b) *$g = \omega z v_+ v_- = \omega' z' v'_- v'_+$, where ω and $\omega' \in \mathcal{N}_{\mathbf{G}}(\mathbf{T})(K)$, z and $z' \in \mathcal{Z}_{\mathbf{G}}(\mathbf{S})(K)$, $v_+ \in \mathbf{V}_\Psi(K)$, $v_- \in \mathbf{V}_\Psi^-(K)$, $v'_+ \in \mathbf{V}_{\Psi'}(K)$ and $v'_- \in \mathbf{V}_{\Psi'}^-(K)$, and*

$$\mathcal{Z}_{\mathbf{S}}(g)^\bullet = \mathcal{Z}_{\mathbf{S}}(v_+)^\bullet = \mathcal{Z}_{\mathbf{S}}(v'_+)^\bullet.$$

Proof. Fix $v \in \mathcal{S}$. We choose $t \in \mathbf{S}(K_v)$ such that $|\alpha(t)|_v \neq 1$ for every root α which is not a linear combination of roots from Λ. Applying Proposition 6.1, we associate to t a system of simple roots Π and a subset Ψ of Π such that $\mathbf{S} = \mathbf{T}_\Psi$, $W^+(t) = \mathbf{V}_\Psi(K_v)$, $W^-(t) = \mathbf{V}_\Psi^-(K_v)$ and $\mathcal{Z}_{\mathbf{G}}(t) = \mathcal{Z}_{\mathbf{G}}(\mathbf{T}_\Psi)$. With these Π and Ψ, let $g = \omega z v_+ v_-$ as given by Lemma 6.4. Now we suppose that t is chosen in such a way that $\dim \mathcal{Z}_{\mathbf{S}}(v_+)$ is minimal. In view of Lemma 6.4(b), it is enough to prove that $\mathcal{Z}_{\mathbf{S}}(v_+)^\bullet \subset \mathcal{Z}_{\mathbf{S}}(v_-)^\bullet$. Suppose by the contrary that $\mathcal{Z}_{\mathbf{S}}(v_+)^\bullet \not\subseteq \mathcal{Z}_{\mathbf{S}}(v_-)^\bullet$. Pick a $t' \in \mathcal{Z}_{\mathbf{S}}(v_+)^\bullet$ such that the subgroup generated by t' is Zariski dense in $\mathcal{Z}_{\mathbf{S}}(v_+)^\bullet$ and for every K-root β either $\beta(t') = 1$ or $|\beta(t)|_v \neq 1$. Then $v_- = w_+ w_0 w_-$ where $w_+ \in W^+(t')$, $w_- \in W^-(t')$ and $w_0 \in \mathcal{Z}_{\mathbf{G}}(t')$. Since $\mathcal{Z}_{\mathbf{S}}(v_+)^\bullet \not\subseteq \mathcal{Z}_{\mathbf{S}}(v_-)^\bullet$ either $w_+ \neq e$ or $w_- \neq e$. Replacing t' by t'^{-1} if necessary, we may suppose that $w_+ \neq e$. Let $\widetilde{t} = t t'^n$ where $n \in \mathbb{N}$. After choosing n sufficiently large, we get that $|\alpha(\widetilde{t})|_v \neq 1$ for every root α which is not a linear combination of roots from Λ, $v_+ w_+ \in W^+(\widetilde{t})$ and

$w_0 w_- \in W^-(\widetilde{t})$. But $\mathcal{Z}_{\mathbf{S}}(v_+ w_+) = \mathcal{Z}_{\mathbf{S}}(v_+) \cap \mathcal{Z}_{\mathbf{S}}(w_+)$. Since $w_+ \neq e$ we obtain that $\dim \mathcal{Z}_{\mathbf{S}}(v_+ w_+) < \dim \mathcal{Z}_{\mathbf{S}}(v_+)$ which contradicts the choice of t. Therefore $\mathcal{Z}_{\mathbf{S}}(v_+)^\bullet \subset \mathcal{Z}_{\mathbf{S}}(v_-)^\bullet$ proving the claim.

The existence of Π' and $\Psi' \subset \Pi'$ as in the formulation of the lemma is proved by virtually the same argument. $\square$

6.2. **Definition of $\mathbf{H}_1$ and reduction of the proof of Theorem 1.8.** Let us define the subgroup $\mathbf{H}_1$ of $\mathbf{G}$ as in the formulation of Theorem 1.8. So, let $g = (g_1, \cdots, g_r) \in G$ be such that $T\pi(g)$ is a locally divergent orbit. Since $g_i \in \mathcal{Z}_{G_i}(T_i)\mathbf{G}(K)$ for all i (Theorem 1.1) the proof is reduced to the case when every $g_i \in \mathbf{G}(K)$.

Next choose $\omega_i \in \mathcal{N}_{\mathbf{G}}(\mathbf{T})(K)$, $1 \leq i \leq r-1$, in such a way that $\dim \bigcap_{i=1}^{r-1} \mathcal{Z}_{\mathbf{T}}(\omega_i g_i g_r^{-1})$ is maximal. Let $\mathbf{H}'_1 = (\mathcal{Z}_{\mathbf{G}}(\bigcap_{i=1}^{r-1} \mathcal{Z}_{\mathbf{T}}(\omega_i g_i g_r^{-1})))^\bullet$. We put

$$\mathbf{H}_1 = g_r^{-1} \mathbf{H}'_1 g_r.$$

It is easy to see that

$$T\pi(g) \subset h_1 H_1 \pi(e),$$

where $h_1 = (\omega_1^{-1} g_r, \cdots, \omega_{r-1}^{-1} g_r, g_r)$ and $H_1 = \mathbf{H}_1(K_{\mathcal{S}})$. Note that $h_1 H_1 \pi(e)$ is closed and T-invariant.

Furthermore, we specify the choice of ω_i as follows. Since

$$\mathbf{H}'_1 = (\mathcal{Z}_{\mathbf{G}}(\bigcap_{i=1}^{r-1} \mathcal{Z}_{\mathbf{T}}(\omega'_i \omega_i g_i g_r^{-1})))^\bullet$$

for all $\omega'_i \in \mathcal{N}_{\mathbf{H}'_1}(\mathbf{T})$, we choose ω_i in such a way that $\dim \mathcal{Z}_{\mathbf{T}}(\omega_i g_i g_r^{-1}) \geq \dim \mathcal{Z}_{\mathbf{T}}(\omega' \omega_i g_i g_r^{-1})$ for all $\omega' \in \mathcal{N}_{\mathbf{H}'_1}(\mathbf{T})$.

Note that $\mathbf{H}_1$ is a reductive K-subgroup, $g_r^{-1} \mathbf{T} g_r \subset \mathbf{H}_1$, $g_r^{-1} \omega_i g_i \in \mathbf{H}_1$ for all i, and

$$T\pi(g) = h_1(g_r^{-1} T g_r)\pi(g_r^{-1}\omega_1 g_1, \cdots, g_r^{-1}\omega_{r-1} g_{r-1}, e).$$

Therefore replacing $\mathbf{G}$ by the quotient of $\mathbf{H}_1$ by its center and $\mathbf{T}$ by the projection of $g_r^{-1} \mathbf{T} g_r$ in this quotient, we reduce the proof of Theorem 1.8 to the following case:

(∗) *all* $g_i \in \mathbf{G}(K)$, $g_r = e$, $\bigcap_i \mathcal{Z}_{\mathbf{T}}(\omega_i g_i)$ *is finite for all choices of* $\omega_i \in \mathcal{N}_{\mathbf{G}}(\mathbf{T})(K)$ *and* $\dim \mathcal{Z}_{\mathbf{T}}(g_i) \geq \dim \mathcal{Z}_{\mathbf{T}}(\omega g_i)$ *for all* $1 \leq i \leq r$ *and all* $\omega \in \mathcal{N}_{\mathbf{G}}(\mathbf{T})(K)$.

Assuming (∗), it is enough to prove that there exists a semisimple subgroup $\mathbf{H}$ of $\mathbf{G}$ and $h \in \mathbf{G}(K)$ such that $\mathrm{rank}_K(\mathbf{G}) = \mathrm{rank}_K(\mathbf{H})$ and

$$\overline{T\pi(g)} \supset hH\pi(e),$$

where $H = \mathbf{H}(K_{\mathcal{S}})$ and $hH\pi(e)$ is T-invariant.

6.3. **Special elements in $\overline{T\pi(g)}$.** In view of the reductions from §6.2, we will suppose up to the end of this and the next sections that the conditions of $(*)$ are fulfilled.

Proposition 6.6. *For every j, $1 \leq j \leq r$, $\overline{T\pi(g)}$ contains an element of the form $\omega(e, \cdots, e, \underset{j}{u}, e, \cdots, e)\pi(h)$, where $h \in \mathbf{G}(K)$, $\omega \in \mathcal{N}_G(T)$, u belongs to a unipotent subgroup of $\mathbf{G}(K)$ normalized by $\mathbf{T}(K)$ and $\mathcal{Z}_{\mathbf{T}}(u)$ is finite.*

Proof. First consider the case when there exists i such that $\mathcal{Z}_{\mathbf{T}}(g_i)$ is finite. Suppose for simplicity that $i = 1$. By Lemma 6.5 and Lemma 6.4 there exists a system of simple roots Π such that every g_i can be writhen in the form $g_i = z_i u_i^+ u_i^-$, where $u_i^+ \in \mathbf{V}_\emptyset(K)$, $u_i^- \in \mathbf{V}_\emptyset^-(K)$, and $z_i \in \mathcal{N}_{\mathbf{G}}(\mathbf{T})(K)$, and, moreover, $\mathcal{Z}_{\mathbf{T}}(u_1^+)$ is finite. Shifting g from the left by an appropriate element from $\mathcal{N}_G(T)$ we may suppose that all $z_i = e$. Since

$$\pi(g) = (g_1(u_{r-1}^-)^{-1}, \cdots, g_{r-2}(u_{r-1}^-)^{-1}, u_{r-1}^+, (u_{r-1}^-)^{-1})\pi(u_{r-1}^-),$$

applying Proposition 6.3, we get

$$(g_1(u_{r-1}^-)^{-1}, \cdots, g_{r-2}(u_{r-1}^-)^{-1}, e, e)\pi(u_{r-1}^-) \in \overline{T\pi(g)}.$$

Repeating the argument $r-2$ times (or using induction on r) we prove that $\overline{T\pi(g)}$ contains an element $(u_1^+ v^-, e, \cdots, e)\pi(h)$, where $h \in \mathbf{G}(K)$ and $v^- \in \mathbf{V}_\emptyset^-(K)$. By Lemma 6.5 there exist opposite minimal parabolic K-subgroups $\widetilde{\mathbf{P}}_\emptyset^-$ and $\widetilde{\mathbf{P}}_\emptyset^-$ containing $\mathbf{T}$ such that $u_1^+ v^- = z w^- w^+$, where $z \in \mathcal{N}_{\mathbf{G}}(\mathbf{T})(K)$, $w^- \in \mathcal{R}_u(\widetilde{\mathbf{P}}_\emptyset^-)(K)$, $w^+ \in \mathcal{R}_u(\widetilde{\mathbf{P}}_\emptyset^+)(K)$ and $\mathcal{Z}_{\mathbf{T}}(w^+)$ is finite. We may suppose that $z = e$. Given $1 < j \leq r$, since

$$(w^- w^+, e, \cdots, e)\pi(h) = (w^- w^+, e, \cdots, e, \underset{j}{(w^+)^{-1}} w^+, e, \cdots, e)\pi(h),$$

Proposition 6.3 implies that $\overline{T\pi(g)}$ contains $(w^+, e, \cdots, \underset{j}{w^+}, \cdots, e)\pi(h)$ and, therefore, it contains $(e, \cdots, \underset{j}{w^+}, \cdots, e)\pi(h)$ too. This completes the proof of the proposition when $\mathcal{Z}_{\mathbf{T}}(g_i)$ is finite for some i.

It easy to see that the proof of the proposition may be reduced to the particular case considered above if we prove that $\overline{T\pi(g)}$ contains an element $\pi(g_1', g_2', g_3, \cdots, g_r)$ such that g_1' and $g_1' \in \mathbf{G}(K)$, $\dim \mathcal{Z}_{\mathbf{T}}(g_i') \geq \dim \mathcal{Z}_{\mathbf{T}}(\omega g_i')$ for all $\omega \in \mathcal{N}_{\mathbf{G}}(\mathbf{T})$, and

$$\mathcal{Z}_{\mathbf{T}}(g_1')^\bullet = \big(\mathcal{Z}_{\mathbf{T}}(g_1) \cap \mathcal{Z}_{\mathbf{T}}(g_2)\big)^\bullet.$$

There is nothing to prove if $\mathcal{Z}_{\mathbf{T}}(g_1)^\bullet \subset \mathcal{Z}_{\mathbf{T}}(g_2)$. Suppose that $\mathcal{Z}_{\mathbf{T}}(g_1)^\bullet \not\subset \mathcal{Z}_{\mathbf{T}}(g_2)$. By Lemmas 6.4 and 6.5 there exist a system of simple roots Π and $\Psi \subset \Pi$ such that $\mathbf{T}_\Psi = \mathcal{Z}_{\mathbf{T}}(g_1)^\bullet$, $g_2 = \omega z v_- v_+$, where $\omega \in \mathcal{N}_{\mathbf{G}}(\mathbf{T})(K)$, $z \in \mathcal{Z}_{\mathbf{G}}(\mathbf{T}_\Psi)(K)$, $v_+ \in \mathbf{V}_\Psi(K)$ and $v_- \in \mathbf{V}_\Psi^-(K)$, and

$$\mathcal{Z}_{\mathbf{T}_\Psi}(g_2)^\bullet = \mathcal{Z}_{\mathbf{T}_\Psi}(v_+)^\bullet.$$

Representing $\pi(g)$ in the form $(g_1(v_+)^{-1}v_+, \omega z v_- v_+, \cdots, g_r)\pi(e)$, Proposition 6.3 implies that $\overline{T\pi(g)}$ contains $(g_1 v_+, \omega z v_+, g_3, \cdots, g_r)\pi(e)$. It is clear that

$$\mathcal{Z}_{\mathbf{T}}(g_1 v_+)^\bullet = (\mathcal{Z}_{\mathbf{T}}(g_1) \cap \mathcal{Z}_{\mathbf{T}}(v_+))^\bullet =$$
$$(\mathcal{Z}_{\mathbf{T}}(g_1) \cap \mathcal{Z}_{\mathbf{T}_\Psi}(g_2))^\bullet = (\mathcal{Z}_{\mathbf{T}}(g_1) \cap \mathcal{Z}_{\mathbf{T}}(g_2))^\bullet,$$

compleating the proof. □

We need the following specification of Proposition 6.6.

Corollary 6.7. *With the notation and assumptions of Proposition 6.6, $\overline{T\pi(g)}$ contains an element of the form $\pi(uh, h, \cdots, h)$, where h and $u \in \mathbf{G}(K)$, u belongs to an abelian unipotent subgroup of $\mathbf{G}$ normalized by $\mathbf{T}$ and $\mathcal{Z}_{\mathbf{T}}(u)$ is finite.*

First we establish the following:

Lemma 6.8. *Consider the $\mathbb{Q}$-vector space $\mathbb{Q}^n$ endowed with the standard scalar product: $((x_1, \cdots, x_n), (y_1, \cdots, y_n)) \overset{def}{=} \sum\limits_i x_i y_i$. Let $v_1, \cdots, v_m$ be pairwise non-proportional vectors in $\mathbb{Q}^n$ and $v \in \mathbb{Q}^n$ be such that $(v_i, v) > 0$ for all $1 \leq i \leq m$. Put $\mathcal{C} = \{\sum\limits_{i=1}^m \alpha_i v_i | a_i \in \mathbb{Q}, a_i \geq 0\}$. Suppose that $m > n$ and the interior of $\mathcal{C}$ with respect to the topology on $\mathbb{Q}^n$ induced by $(\cdot, \cdot)$ is not empty. Then there exist $1 \leq i_\circ \leq m$ and $w \in \mathbb{Q}^n$ such that $(w, v_{i_\circ}) < 0$, $(w, v_i) > 0$ if $i \neq i_\circ$ and $\{v_i | i \neq i_\circ\}$ contains a basis of $\mathbb{Q}^n$.*

Proof. Let $\mathbb{Q}v_i \cap \mathcal{C}$, $1 \leq i \leq m_1$, be the edges of the cone $\mathcal{C}$. Then $m_1 \geq n$ and $v_1, \cdots, v_n$ is a bases of $\mathbb{Q}^n$. If $m_1 = n$ and $v_{n+1} = \sum\limits_{i=1}^n c_i v_i$ one of $c_i > 0$. We suppose that $c_1 > 0$. Let

$$\mathcal{C}' = \{\sum_{i=2}^m a_i v_i | a_i \in \mathbb{Q}, a_i \geq 0\}.$$

It is easy to see that in both cases $m_1 > n$ and $m_1 = n$ the interior of the cone $\mathcal{C}'$ is nonempty and $\mathcal{C}'$ contains all v_i but v_1. Therefore there exists $w \in \mathbb{Q}^n$ such that $(w, v_1) < 0$ and $(w, v_i) > 0$ for all $i > 1$. □

Proof of Corollary 6.7. Let $\mathbf{V}$ be the minimal $\mathbf{T}$-invariant unipotent K-subgroup of $\mathbf{G}$ containing u. There exists a system of positive roots Φ^+ such that the corresponding to Φ^+ maximal unipotent K-subgroup contains $\mathbf{V}$. Let Φ^+_{nd} be the set of non-divisible roots and $\{\alpha_1, \cdots, \alpha_m\} = \{\alpha \in \Phi^+_{nd} : \mathbf{U}_{(\alpha)} \cap \mathbf{V} \neq \{e\}\}$ where $\mathbf{U}_{(\alpha)}$ is the corresponding to α root group. Put $\mathbf{V}_{\alpha_i} = \mathbf{U}_{(\alpha_i)} \cap \mathbf{V}$. Then $\mathbf{V}$ is directly spanned by $\mathbf{V}_{\alpha_i}$ taken in any order (cf.[B, 21.9]). It follows from the minimality assumption in the definition of $\mathbf{V}$ and the fact that every $(\alpha_i) = \{\alpha_i\}$ or $\{\alpha_i, 2\alpha_i\}$ that all $\mathbf{V}_{\alpha_i}$ are abelian.

We will complete the proof by induction on $\dim \mathbf{V}$. There is nothing to prove if $\mathbf{V}$ is abelian. Suppose that the derived subgroup $\mathfrak{D}(\mathbf{V})$ of $\mathbf{V}$ is not trivial.

Let $u = u_1 \cdots u_m$ where $u_i \in \mathbf{V}_i(K)$. There exists $1 \leq l \leq m$ such that after a rearrangement of $\{\alpha_1, \cdots, \alpha_m\}$ we have $u_i \notin \mathfrak{D}(\mathbf{V})$ if and only if $1 \leq i \leq l$. Then every $\alpha_j, j > l$, is a linear combination with positive coefficients of at least two roots in $\{\alpha_1, \cdots, \alpha_l\}$. Since $\mathcal{Z}_{\mathbf{T}}(u)$ is finite and $\mathbf{V}$ is not abelian, $\{\alpha_1, \cdots, \alpha_l\}$ contains a basis of the $\mathbb{Q}$-vector space $X(\mathbf{T})\bigotimes_{\mathbb{Z}}\mathbb{Q}$ and $m > l$. Put $\mathcal{C} = \{\sum_{i=1}^{m} a_i\alpha_i | a_i \in \mathbb{Q}, a_i \geq 0\}$. By Lemma 6.8 there exist $1 \leq i \leq l$, say $i = 1$, and $t \in \mathbf{T}(K)$ such that $\lim_{n\to+\infty} t^n u_1 t^{-n} = e$ in G_1 and $\lim_{n\to-\infty} t^n u_i t^{-n} = e$ in G_1 for all $i > 1$. Put $u' = u_2 \cdots u_m$. It follows from Proposition 6.3 that $\overline{T\pi(g)}$ contains $\pi(u'h, h, \cdots, h)$. Since $\mathcal{Z}_{\mathbf{T}}(u')$ is finite and u' is contained in a proper $\mathbf{T}$-invariant K-subgroup of $\mathbf{V}$ the corollary is proved. □

6.4. **Unipotent orbits on $\overline{T\pi(g)}$.** Further on some propositions formulated in $\mathcal{S}$-adic setting will be deduced from their archimedean analogs when $\mathcal{S} = \mathcal{S}_\infty$. For this purpose the following lemma is needed.

Lemma 6.9. *Let $\mathbf{V}$ be a unipotent K-algebraic group and $\mathbf{U}$ be its K-subgroup. Put $U = \mathbf{U}(K_\mathcal{S})$ and $U_\infty = \mathbf{U}(K_\infty)$. Let M be a subset of U_∞ such that $\overline{M\mathbf{V}(\mathcal{O}_\infty)} = U_\infty\mathbf{V}(\mathcal{O}_\infty)$. Then*

$$\overline{M\mathbf{V}(\mathcal{O})} = U\mathbf{V}(\mathcal{O}).$$

Proof. By the strong approximation for unipotent groups (see, for example, [PR, §7.1, Corollary]), we have that $U = \overline{U_\infty\mathbf{U}(\mathcal{O})}$. Using that $\mathbf{U}(\mathcal{O}) \subset \mathbf{V}(\mathcal{O})$, $\mathbf{V}(\mathcal{O}_\infty) \subset \mathbf{V}(\mathcal{O})$ and $U\mathbf{V}(\mathcal{O})$ is closed, we get

$$U\mathbf{V}(\mathcal{O}) = \overline{U_\infty\mathbf{V}(\mathcal{O})} = \overline{U_\infty\mathbf{V}(\mathcal{O}_\infty)\mathbf{V}(\mathcal{O})} = \overline{M\mathbf{V}(\mathcal{O}_\infty)\mathbf{V}(\mathcal{O})} = \overline{M\mathbf{V}(\mathcal{O})}.$$

□

Proposition 6.10. *We suppose that $r > 2$, K is not a* CM*-field and the completion K_1 is archimedean. Let $\mathbf{V}$ be an abelian unipotent K-subgroup of $\mathbf{G}$ normalized by $\mathbf{T}$. Let $u \in \mathbf{V}(K)$ and $\mathcal{Z}_{\mathbf{T}}(u)$ be finite. Then there exists a K-subgroup $\mathbf{U}$ of $\mathbf{V}$ which is $\mathbf{T}$-invariant, contains u, and*

$$U\pi(e) = \overline{\mathbf{T}(\mathcal{O})(u, e, \cdots, e)\pi(e)} = \overline{\{(tut^{-1}, e, \cdots, e)\pi(e) : t \in \mathbf{T}(\mathcal{O})\}}, \tag{15}$$

where $U = \mathbf{U}(K_\mathcal{S})$

As in §3, we denote by L the closure of the projection of $\mathcal{O}^*$ in K_1^*. There are two possibilities: either $L = K_1^*$ or $L \neq K_1^*$. Since K is not a CM-field, Proposition 3.2 implies that in the latter case $K_1 = \mathbb{C}$, $\dim L = 1$ and $L \neq \mathbb{R}_+$.

Proof of Proposition 6.10 when $L = K_1^*$. Since $\mathbf{V}$ is normalized by $\mathbf{T}$ there exists an order Φ^+ of the set of K-roots with respect to $\mathbf{T}$ such that $\mathbf{V} \subset \mathbf{V}_\emptyset$. Therefore, identifying $\mathbf{T}(K_1)$ with $(K_1^*)^{\dim \mathbf{T}}$, the map $\mathbf{T}(K_1) \to \mathbf{V}(K_1), t \mapsto tut^{-1}$, coincides with the restriction to $(K_1^*)^{\dim \mathbf{T}}$ of a polynomial map $K_1^{\dim \mathbf{T}} \to \mathbf{V}(K_1)$. Let $\pi_\infty : V_\infty \mapsto V_\infty/\mathbf{V}(\mathcal{O}_\infty)$, where $V_\infty = \mathbf{V}(K_\infty)$,

be the natural projection. By the polynomial measure rigidity for tori (cf. [Wey] or [Sh, Corollary 1.2] for a more general result) there exists a $\mathbf{T}$-invariant K-subgroup $\mathbf{U}$ of $\mathbf{V}$ such that

$$U_\infty \pi_\infty(e) = \overline{\{(tut^{-1}, e, \cdots, e)\pi_\infty(e) : t \in \mathbf{T}(K_1)\}},$$

where $U_\infty = \mathbf{U}(K_\infty)$. Now (15) follows from Lemma 6.9. $\square$

Proof of Proposition 6.10 when $K_1 = \mathbb{C}$ and $\dim L = 1$. Up to a subgroup of finite index there are two possibilities for L: there exists $\alpha \in \mathbb{R}^*$ such that either (a) L is a direct product of the unit circle group S^1 and an infinite cyclic group, i.e. $L = \{e^{2\pi n\alpha + \mathrm{i}t} : n \in \mathbb{Z}, 0 \leq t < 2\pi\}$, where $\mathrm{i}^2 = -1$, or (b) L is a spiral, i.e. $L = \{e^{(\alpha+\mathrm{i})(t+2\pi n)} : n \in \mathbb{Z}, 0 \leq t < 2\pi\}$. The case $\alpha < 0$ being analogous to the case $\alpha > 0$, further on we will suppose that $\alpha > 0$. In order to treat the cases (a) and (b) simultaneously, note that $L = \{e^{2\pi n\alpha + (\widetilde{\alpha}+\mathrm{i})t} : n \in \mathbb{Z}, 0 \leq t < 2\pi\}$ where $\widetilde{\alpha} = 0$ in case (a) and $\widetilde{\alpha} = \alpha$ in case (b). (We use the equality $e^{2\pi n\alpha + (\alpha+\mathrm{i})t} = e^{(\alpha+\mathrm{i})(t+2\pi n)}$.)

Given $\theta \in [0, 2\pi)$ and $b < c$, we denote $[b, c]_\theta = \{re^{\mathrm{i}\varphi} : a < r < b\}$ and $\mathbb{R}_\theta = \{re^{\mathrm{i}\theta} : r \in \mathbb{R}\}$. As usual, $\mathbf{GL}_1$ stands for the 1-dimensional $\mathbb{Q}$-split torus.

Returning to the proof of Proposition 6.10, note that $\mathbf{V}$ is a K-vector space and $\mathbf{V} = \bigoplus_i^l \mathbf{V}_{\lambda_i}$ where $\mathbf{V}_{\lambda_i}$ are different weight spaces for the action of $\mathbf{T}$ on $\mathbf{V}$. Since $u \in \mathbf{V}(K)$ we have $u = \sum_i u_i$ where $u_i \in \mathbf{V}_{\lambda_i}(K)$. Next using that the projection of $\mathcal{O}$ into $\prod_{v \in \mathcal{S} \setminus \{v_1\}} K_v$ is dense, we get that $\mathbf{U}$ as in the formulation of the proposition is spanned by the vectors u_i. This allows to replace $\mathbf{T}$ by its one dimensional sub-torus $\mathbf{T}'$ such that the restrictions of λ_i to $\mathbf{T}'$ are pairwise different. Therefore Proposition 6.10 follows immediately from the following:

Lemma 6.11. *Let $\mathbf{GL}_1$ act K-rationally on a finite dimensional K-vector space $\mathbf{V}$ and $\mathbf{V} = \bigoplus_i^l \mathbf{V}_{\lambda_i}$ be the decomposition of $\mathbf{V}$ as a sum of one-dimensional weight sub-spaces with weights $\lambda_i(t) = t^{n_i}$. Suppose that $r > 2$, K is not a* CM*-field, and n_i are pairwise different positive integers. Let $u = \sum_i u_i$ where $u_i \in \mathbf{V}_{\lambda_i}(K) \setminus \{0\}$ for all i. Then for every real $C > 1$, we have*

$$V = \overline{\{(\sum_i \lambda_i(a)u_i, 0, \cdots, 0) + \mathbf{V}(\mathcal{O}) : a \in L, |a|_1 \geq C\}},$$

where $V = \mathbf{V}(K_\mathcal{S})$.

Proof. We will (as we may) suppose that $\mathbf{V}(K) = K^l$ and $u_i = (0, \cdots, \underset{i}{1}, \cdots, 0)$ for all i. Since the projection of K_1 into $K_\mathcal{S}/\mathcal{O}$ is dense, it follows from [Sh,

Corollary 1.2] (or [Wey]) that for every $C > 1$

$$K^l_{\mathcal{S}} = \overline{\{(\sum_i \lambda_i(a), 0, \cdots, 0) + \mathcal{O}^l : a \in K_1, |a|_1 \geq C\}}.$$

In view of Lemma 6.9, we need to prove that

$$V_\infty = \overline{\{(\sum_i \lambda_i(a)u_i, 0, \cdots, 0) + \mathbf{V}(\mathcal{O}_\infty) : a \in L, |a|_1 \geq C\}},$$

where $V_\infty = \mathbf{V}(K_\infty)$.

Let $0 < n_1 < n_2 < \cdots < n_l$. For every i we introduce a parametric curve $f_i : [0, 2\pi) \to \mathbb{C}^*_1$, $t \mapsto e^{(\widetilde{\alpha}+\mathrm{i})n_i t}$. Since $\mathcal{O}_\infty$ is a group of finite type and $\overline{\mathbb{R}_\theta + \mathcal{O}_\infty}$, $0 \leq \theta < 2\pi$, is a subspace of the real vector space K_∞, the set of all $0 \leq \theta < 2\pi$ such that $\overline{\mathbb{R}_\theta + \mathcal{O}_\infty} \subsetneq K_\infty$ is countable. The tangent line at t of the curve $f_i(t)$ runs over all directions when $0 \leq t < 2\pi$. Therefore there exist $0 \leq \psi < 2\pi$ such that if the tangent line at ψ of the parametric curve f_i is parallel to $\mathbb{R}_{\theta_i}$, $0 \leq \theta_i < 2\pi$, then $\overline{\mathbb{R}_{\theta_i} + \mathcal{O}_\infty} = K_\infty$ for all $1 \leq i \leq l$.

For every $n \in \mathbb{N}_+$, let

$$F_n : [0, 2\pi) \to K^l_\infty, \quad t \mapsto ((e^{2\pi n n_1 \alpha} f_1(t), \cdots, 0), \cdots, (e^{2\pi n n_l \alpha} f_l(t), \cdots, 0)).$$

A subset M of $K^l_\infty/\mathcal{O}^l_\infty$ will be called *ε-dense* if the ε-neighborhood of any point in $K^l_\infty/\mathcal{O}^l_\infty$ contains an element from M. (As usual, $K^l_\infty/\mathcal{O}^l_\infty$ is endowed with a metrics induced by the standard metrics on K_∞ considered as a real vector space.)

Now the lemma follows from the next

Claim. *With ψ and F_n as above, let $\varepsilon > 0$. There exist reals $A_\varepsilon > 0$ and $b_\varepsilon > 0$ such that if $\psi - b_\varepsilon \leq c < d \leq \psi + b_\varepsilon$ and $e^{2\pi n n_1}(d - c) > A_\varepsilon$ for some c and $d \in \mathbb{R}$ and $n \in \mathbb{N}_+$, then*

$$\{F_n(t) + \mathcal{O}^l_\infty | c \leq t \leq d\}$$

is ε-dense in $K^l_\infty/\mathcal{O}^l_\infty$.

We will prove the claim by induction on l. Let $l = 1$, i.e. $F_n : [0, 2\pi) \to K_\infty$, $t \mapsto (e^{2\pi n n_1 \alpha} f_1(t), 0, \cdots, 0)$, where $f_1(t) = e^{(\widetilde{\alpha}+\mathrm{i})n_1 t}$. It follows from the choice of ψ that there exists a real $B_\varepsilon > 0$ such that the projection of $[0, B_\varepsilon]_{\theta_1}$ into $K_\infty/\mathcal{O}_\infty$ is $\frac{\varepsilon}{2}$-dense. Hence every shift of $[0, B_\varepsilon]_{\theta_1} + \mathcal{O}_\infty$ by an element from K_∞ is $\frac{\varepsilon}{2}$-dense in $K_\infty/\mathcal{O}_\infty$ too. Choosing A_ε sufficiently large and b_ε sufficiently small we get that if n is such that $e^{2\pi n n_1}(2b_\varepsilon) > A_\varepsilon$ then the length of the curve $\{F_n(t) | \psi - b_\varepsilon \leq t \leq \psi + b_\varepsilon\}$ is greater than B_ε and if I is any connected piece of this curve of length B_ε then I is $\frac{\varepsilon}{2}$-close (with respect to the Hausdorff metrics on $\mathbb{C}$) to a shift of $[0, B_\varepsilon]_{\theta_1}$. This implies the claim for $l = 1$.

Now suppose that $l > 1$ and the claim is valid for $l - 1$. Let

$$\widetilde{F}_n(t) = ((e^{2\pi n n_1 \alpha} f_1(t), \cdots, 0), \cdots, (e^{2\pi n n_{l-1} \alpha} f_{l-1}(t), \cdots, 0)).$$

It follows from the induction hypothesis for $l-1$ and from the validity of the Claim for $l=1$ that for every $\varepsilon > 0$ there exist positive reals A_ε and b_ε such that if $\psi - b_\varepsilon \leq c < d \leq \psi + b_\varepsilon$ and $e^{2\pi n n_1}(d-c) \geq A_\varepsilon$ for some $n \in \mathbb{N}_+$ then $\{\widetilde{F}_n(t) + \mathcal{O}_\infty^{l-1} | c \leq t \leq d\}$ is ε-dense in $K_\infty^{l-1}/\mathcal{O}_\infty^{l-1}$ and

$$\{(e^{2\pi n n_l \alpha} f_l(t), 0 \cdots, 0) + \mathcal{O}_\infty | c' \leq t \leq d'\}$$

is ε-dense in $K_\infty/\mathcal{O}_\infty$ whenever $c < c' < d' < d$ and $e^{2\pi n n_l}(d'-c') \geq A_\varepsilon$.

Further on, given $c_* < d_*$, we define the length of the parametric curve $\{\widetilde{F}_n(t) | c_* \leq t \leq d_*\} \subset K_\infty^{l-1}$ as the maximum of the lengths of the curves $\{e^{2\pi n n_i \alpha} f_i(t) | c_* \leq t \leq d_*\} \subset \mathbb{C}$, $1 \leq i \leq l-1$. With b_ε, A_ε and n as above, let $x \in K_\infty^{l-1}/\mathcal{O}_\infty^{l-1}$. There exist $c_{x,n}$ and $d_{x,n}$ such that $c < c_{x,n} < d_{x,n} < d$ and $\{\widetilde{F}_n(t) + \mathcal{O}_\infty^{l-1} | c_{x,n} \leq t \leq d_{x,n}\}$ is of length $\frac{\varepsilon}{2}$ and contained in an ε-neighborhood of x. In view of the definition of $\widetilde{F}_n$, there exists δ not depending on x and n such that $e^{2\pi n n_{l-1}}(d_{x,n} - c_{x,n}) \geq \delta$. Now, since $n_l > n_{l-1}$, choosing n large enough we get that $e^{2\pi n n_l}(d_{x,n} - c_{x,n}) \geq A_\varepsilon$ completing the proof of the claim. □

6.5. **A refinement of Jacobson-Morozov lemma.** We will need the following known lemma (cf.[E-L, Lemma 3.1]):

Lemma 6.12. *Let $\mathbf{L}$ be a semisimple group over a field F of characteristic 0, $\mathbf{T}$ be a maximal F-split torus in $\mathbf{L}$, α be an indivisible root with respect to $\mathbf{T}$ and $\mathbf{U}_{(\alpha)}$ be the corresponding to α root group. Denote*

$$U = \Big\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in F \Big\},\ D = \Big\{ \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} : b \in F^* \Big\}.$$

Let $u \in \mathbf{U}_{(\alpha)}(F)$. Suppose that $u = \exp(v)$ where v belongs to the root space $\mathfrak{g}_\alpha$ or $\mathfrak{g}_{2\alpha}$. Then there exists an F-morphism $f : \mathbf{SL}_2 \to \mathbf{L}$ such that $u \in f(U)$ and $f(D) \subset \mathbf{T}(F)$.

6.6. **Actions of epimorphic subgroups on homogeneous spaces in $\mathcal{S}$-adic setting.** Recall that $\mathbf{G}$ is a K-isotropic semisimple K-group and $\mathcal{S} \supset \mathcal{S}_\infty$. We have $G = G_\infty \times G_f$ where $G_\infty = \prod_{v \in \mathcal{S}_\infty} G_v$ and $G_f = \prod_{v \in \mathcal{S}_f} G_v$. Let H be a closed subgroup of G_∞ which have finite index in its Zariski closure. Recall that a subgroup B of H is called *epimorphic* if all B-fixed vectors are H-fixed for every rational linear representation of H. For example, the parabolic subgroups in H are epimorphic.

In the case when $\mathcal{S} = \mathcal{S}_\infty$ the following proposition is proved in [Sh-W, Theorem 1].

Proposition 6.13. *Let H be a subgroup of G_∞ generated by 1-parameter unipotent subgroups and B be an epimorphic subgroup of H. Then any closed B-invariant subset of G/Γ is H-invariant.*

Proof. We need to prove that $\overline{B\pi(g)}$ is H-invariant for every $g \in G$. Since $g^{-1}Bg$ is an epimorphic subgroup of $g^{-1}Hg$ it is enough to prove that $\overline{B\pi(e)}$

is H-invariant, that is, $\overline{B\pi(e)} = \overline{H\pi(e)}$. Let $G_{f,n}$ be a decreasing sequence of compact subgroups of G_f such that $\bigcap_n G_{f,n} = \{e\}$. Let $G_n = G_\infty \times G_{f,n}$ and $\Gamma_n = \Gamma \cap G_n$. Let $\phi_n : G \to G_\infty$ be the natural projection and $\Gamma_{n,\infty} = \phi_n(\Gamma_n)$.

In view of [Sh-W, Theorem 1]

$$\overline{B\Gamma_{n,\infty}} = \overline{H\Gamma_{n,\infty}}.$$

By the topological rigidity for unipotent groups [Ra1], for every n there exists a connected subgroup L_n of G_∞ which contains H and

$$\overline{H\Gamma_{n,\infty}} = L_n\Gamma_{n,\infty}.$$

But $L_n \cap \Gamma_{n,\infty}$ is a lattice in L_n and Γ_n has finite index in Γ_{n+1}. It follows from the connectedness of L_n that all L_n coincide, i.e. $L_n = L$. So, $\phi_n(\overline{B\Gamma_n}) = \overline{B\Gamma_{n,\infty}} = L\Gamma_{n,\infty}$. Hence for every $x \in L$ there exists $a_n \in G_{f,n}$ such that $xa_n \in \overline{B\Gamma_n}$. Since xa_n converges to x in G, we get that $x \in \overline{B\Gamma}$, i.e. $L \subset \overline{B\Gamma}$. In view of the inclusions $B \subset H \subset L$ it is obvious that

$$\overline{B\Gamma} = \overline{H\Gamma},$$

proving the proposition. $\square$

6.7. **Proof of Theorem 1.8.** We keep the assumptions from §6.2. Shifting $\pi(g)$ from the left by an appropriate element from $\mathcal{N}_G(T)$ we may suppose, in view of Propositions 6.6 and 6.10, that there exists a unipotent subgroup $\mathbf{U}$ defined over K and normalized by $\mathbf{T}$ such that $\mathcal{Z}_{\mathbf{T}}(\mathbf{U})$ is finite and $\overline{T\pi(g)} \supset U\pi(h)$ where $h \in \mathbf{G}(K)$ and $U = \mathbf{U}(K_{\mathcal{S}})$. Let $\mathbf{H}$ be a Zariski connected K-subgroup of $\mathbf{G}$ with the properties: $\mathbf{U} \cup \mathbf{T} \subset \mathbf{H}$ and $H\pi(h) \subset \overline{T\pi(e)}$ where $H = \mathbf{H}(K_{\mathcal{S}})$. We choose $\mathbf{H}$ to be maximal with the above properties. Let $u \in (\mathbf{U}_{(\alpha)} \cap \mathbf{H})(K)$, $u \neq e$, and $u = \exp(v)$ where $\alpha \in \Phi^+_{nd}$ and $v \in \mathfrak{g}_\alpha \cup \mathfrak{g}_{2\alpha}$. By Lemma 6.12 there exists a K-morphism $f : \mathbf{SL}_2 \to \mathbf{G}$ such that if $\mathbf{B}$ is the group of upper triangular matrices in $\mathbf{SL}_2(K)$ then $u \in f(\mathbf{B})$ and $f(\mathbf{B}) \subset \mathbf{H}$. Using Proposition 6.13 we conclude that $\overline{H\pi(h)}$ is invariant under the action of the subgroup spanned by $f(\mathbf{SL}_2(K_{\mathcal{S}}))$ and H. In view of the maximality in the choice of $\mathbf{H}$ we get that $f(\mathbf{SL}_2) \subset \mathbf{H}$. Therefore $\mathbf{H}$ is a reductive subgroup of maximal K-rank. Since $\mathbf{U} \subset \mathbf{H}$ and $\mathcal{Z}_{\mathbf{T}}(\mathbf{U})$ is finite, $\mathbf{H}$ is semisimple. $\square$

6.8. **Proof of Corollary 1.9.** We suppose that $\mathbf{G} = \mathbf{SL}_{n+1}, n \geq 1$. As usual, $\mathbf{SL}_{n+1} = \mathbf{SL}(\mathbf{W})$ where $\mathbf{W}$ is the standard K-vector space with $\mathbf{W}(K) = K^{n+1}$ and $\mathbf{W}(\mathcal{O}) = \mathcal{O}^{n+1}$.

Corollary 1.9 follows immediately from the next proposition.

Proposition 6.14. *Let $\mathbf{H}$ be a proper Zariski connected reductive K-subgroup of $\mathbf{SL}_{n+1}$ of K-rank n. Then $\mathbf{H}$ is not semisimple and, moreover, the following holds:*

(a) *there exists a direct sum decomposition* $\mathbf{W} = \mathbf{W}_1 \oplus \cdots \oplus \mathbf{W}_r$, *where* $r \geq 2$ *and* $\mathbf{W}_i$ *are non-zero* K*-subspaces of* $\mathbf{W}$, *such that* $\mathbf{H} = \{g \in \mathbf{SL}_{n+1} : g\mathbf{W}_i = \mathbf{W}_i \text{ for all } i\}$;

(b) *if* $\mathbf{H}'$ *is a reductive* K*-subgroup of* $\mathbf{SL}_{n+1}$ *such that* $\mathbf{H}' \supset \mathbf{H}$ *and* $\text{s.s.rank}_K(\mathbf{H}) = \text{s.s.rank}_K(\mathbf{H}')$ *then* $\mathbf{H} = \mathbf{H}'$.

Proof. By Borel-De Siebenthal theory [BD] every maximal connected subgroup of $\mathbf{SL}_{n+1}$ containing $\mathbf{H}$ has a one dimensional center. Therefore the (Zariski) connected component $\mathbf{Z}$ of the center of $\mathbf{H}$ is not trivial. Let $\mathbf{W} = \mathbf{W}_1 \oplus \cdots \oplus \mathbf{W}_r$, $r \geq 2$, where $\mathbf{W}_i$ are the weight subspaces for the action of $\mathbf{Z}$ on $\mathbf{W}$. Each $\mathbf{W}_i$ is $\mathbf{H}$-invariant and $\mathbf{H}$ is an almost direct product of $\mathbf{Z}$ and $\mathbf{SL}(\mathbf{W}_1) \cap \mathbf{H} \times \cdots \times \mathbf{SL}(\mathbf{W}_r) \cap \mathbf{H}$. Since $\mathbf{H}$ has maximal K-rank we get that the K-rank of $\mathbf{SL}(\mathbf{W}_i) \cap \mathbf{H}$ is equal to $\dim \mathbf{W}_i - 1$ for every i. Therefore $\mathbf{SL}(\mathbf{W}_i) \subset \mathbf{H}$. This implies that $\mathbf{H} = \{g \in \mathbf{SL}_{n+1} : g\mathbf{W}_i = \mathbf{W}_i \text{ for all } i\}$.

In order to prove (b) note that $\mathbf{H} = \mathcal{Z}_{\mathbf{G}}(\mathbf{Z})$ and $\mathfrak{D}(\mathbf{H}) = \mathbf{SL}(\mathbf{W}_1) \times \cdots \times \mathbf{SL}(\mathbf{W}_r)$. Hence if $\mathbf{H}'$ is a reductive K-subgroup of $\mathbf{SL}_{n+1}$ containing $\mathbf{H}$ and $\text{s.s.rank}_K(\mathbf{H}) = \text{s.s.rank}_K(\mathbf{H}')$ then $\mathbf{H} = \mathbf{H}'$. □

7. A number theoretical application

7.1. **Reduction of the proof of Theorem 1.10 to the case** $m = n$**.** We need the following simple

Proposition 7.1. *Let* $M_1, \cdots, M_r$ *be finite subsets of the vector space* K^n *with the following properties: each* M_i *consists of* m *linearly independent vectors and there exist* $w \in M_1$ *and* $j \geq 2$ *such that no vector from* M_j *is proportional to* w. *Then there exists a linear map* $\phi : K^n \to K^m$ *such that every* $\phi(M_i)$ *consists of* m *linearly independent vectors and no vector from* $\phi(M_j)$ *is proportional to* $\phi(w)$.

Proof. It easy to see that the conditions on ϕ define a non-empty Zariski open subset of the vector space of all linear maps from K^n to K^m. This implies the proposition. □

In order to reduce the proof of Theorem 1.10 to the case $m = n$ we apply Proposition 7.1 to the vector space $Kx_1 + \cdots + Kx_n$ and the subsets $M_v = \{l_1^{(v)}(\vec{x}), \ldots, l_m^{(v)}(\vec{x})\}$, $v \in \mathcal{S}$. There exists a basis $\vec{y} = (y_1, \ldots, y_n)$ of $Kx_1 + \cdots + Kx_n$ such that the map ϕ, as in the formulation of Proposition 7.1, is given by $\phi(a_1y_1 + \cdots + a_ny_n) = a_1y_1 + \cdots + a_my_m$. Let $l_i^{(v)}(\vec{x}) = \widetilde{l}_i^{(v)}(\vec{y})$ for all i and v. Then the linear forms $\widetilde{l}_1^{(v)}(y_1, \cdots, y_m, 0, \cdots, 0), \cdots, \widetilde{l}_m^{(v)}(y_1, \cdots, y_m, 0, \cdots, 0)$ are linearly independent over K and the polynomials $\prod_{i=1}^{m} \widetilde{l}_i^{(v)}(y_1, \cdots, y_m, 0, \cdots, 0)$, $v \in \mathcal{S}$, are not pairwise proportional. This completes the reduction.

7.2. **Proof of Theorem 1.10.** Let $\mathbf{G} = \mathbf{SL}_{n+1}$, $G = \mathbf{SL}_{n+1}(K_{\mathcal{S}})$ and $\Gamma = \mathbf{SL}_{n+1}(\mathcal{O})$. The group G is acting on $K_{\mathcal{S}}[\vec{x}]$ according to the law $(\sigma\phi)(\vec{x}) = \phi(\sigma^{-1}\vec{x})$, where $\sigma \in G$ and $\phi \in K_{\mathcal{S}}[\vec{x}]$. We denote $f_0(\vec{x}) = x_1 x_2 ... x_{n+1}$. Let $f(\vec{x})$ be as in the formulation of the theorem and $m = n$. There exists $g = (g_v)_{v\in\mathcal{S}} \in G$ such that every $g_v \in \mathbf{G}(K)$ and $f(\vec{x}) = \alpha(g^{-1}f_0)(\vec{x})$ where $\alpha \in K_{\mathcal{S}}$. Since $f_v(\vec{x}), v_n \in S$, are not pairwise proportional the orbit $T\pi(g)$ is locally divergent but non-closed (Theorem 1.1). Note that $f(\vec{x}) = \alpha(wgf_0)(\vec{x})$ for every $w \in \mathcal{N}_G(T)$. In view of Theorem 1.8 and its Corollary 1.9 there exist a reductive K-subgroup $\mathbf{H}$ with $\mathbf{T} \subsetneq \mathbf{H}$ and $\sigma \in \mathbf{G}(K)$ such that $\overline{T\pi(g)} = H\pi(\sigma)$ where $H = \mathbf{H}(K_{\mathcal{S}})$. By Proposition 6.14 there exists a direct sum decomposition $\mathbf{W} = \mathbf{W}_1 \oplus \cdots \oplus \mathbf{W}_r$ such that $\mathbf{H} = \{g \in \mathbf{SL}_{n+1} : g\mathbf{W}_i = \mathbf{W}_i \text{ for all } i\}$. (We use that $\mathbf{SL}_{n+1}$ is identified with $\mathbf{SL}(\mathbf{W})$.) Therefore given $a = (a_v)_{v\in\mathcal{S}}$ there exist $\vec{z} \in \mathcal{O}^{n+1}$ and $h = (h_v)_{v\in\mathcal{S}} \in H$ such that $f_0(h\sigma(\vec{z})) = a$. Since $H\sigma\Gamma = \overline{Tg\Gamma}$ there exist $t_i \in T$ and $\gamma_i \in \Gamma$ such that

$$\lim_i t_i g \gamma_i = h\sigma.$$

Therefore

$$\lim_i f(\gamma_i \vec{z}) = a,$$

proving the theorem. □

8. Examples

The goal of this section is to show that the hypothesis in the formulations of Theorem 1.8, Corollary 1.9 and Theorem 1.10 are essential and can not be removed. Up to the end of §8 we will suppose that $\mathcal{S}$ consists of archimedean valuations and K is a CM-field. We have $K = F(\sqrt{-d})$, where F is a totally real field, $d \in F$ and $d > 0$ in every archimedean completion of F. With slight abuse of notation, we will denote in the way the archimedean valuations of F and their (unique) extensions to K. So, $K_v = \mathbb{C}$ and $F_v = \mathbb{R}$ for every $v \in \mathcal{S}$. We denote by $\mathcal{O}_F$ (resp. $\mathcal{O}_K$) the ring of integers of F (resp. K). Recall that $\mathcal{O}_F$ (resp. $\mathcal{O}_K$) is a lattice in $F_{\mathcal{S}} = \prod_{v\in\mathcal{S}} F_v$ (resp. $K_{\mathcal{S}} = \prod_{v\in\mathcal{S}} K_v$).

8.1. **Restriction of scalars functor for** CM**-fields.** Denote by $\mathbf{G}$ the group $\mathbf{SL}_2$ considered as a K-algebraic group. Let $^{-} : K \to K$ be the non-trivial automorphism of K/F. For every $v \in \mathcal{S}$ we keep the same notation $^{-}$ for the complex conjugation of $K_v = \mathbb{C}$ and for the group automorphism $\mathrm{SL}_2(K_v) \to \mathrm{SL}_2(K_v), \begin{pmatrix} x & y \\ z & t \end{pmatrix} \mapsto \begin{pmatrix} \overline{x} & \overline{y} \\ \overline{z} & \overline{t} \end{pmatrix}$. There exists a simple F-algebraic group of F-rank 1 denoted by $R_{K/F}(\mathbf{G})$ and a K-morphism $p : R_{K/F}(\mathbf{G}) \to \mathbf{G}$ such that the map $(p, \overline{p}) : R_{K/F}(\mathbf{G}) \to \mathbf{G} \times \mathbf{G}, g \mapsto (p(g), \overline{p(g)})$, is a K-isomorphism of K-algebraic groups and $p(R_{K/F}(\mathbf{G})(F)) = \mathbf{G}(K)$. Note that the map $R_{K/F}(\mathbf{G})(F) \to \mathbf{G}(K), g \mapsto p(g)$, is a group isomorphism. The pair

$(R_{K/F}(\mathbf{G}), p)$ is uniquely defined by the above properties up to F-isomorphism and one says that the F-algebraic group $R_{K/F}(\mathbf{G})$ is obtained from the K-algebraic group $\mathbf{G}$ by restriction scalars from K to F. (We refer to [BT, 6.17-6.21] or [W2, 1.3] for the general definition and basic properties of the restriction of scalars functor $R_{K/F}(\cdot)$.)

Given $v \in \mathcal{S}$, the isomorphism $R_{K/F}(\mathbf{G})(F) \to \mathbf{G}(K), g \mapsto p(g)$ admits a unique extension to an isomorphism $R_{K/F}(\mathbf{G})(F_v) \to G_v$ denoted by p_v. Let $p_{\mathcal{S}}$ be the direct product of all $p_v, v \in \mathcal{S}$. Further on $R_{K/F}(\mathbf{G})(F_{\mathcal{S}})$ will be identified with G via the isomorphism $p_{\mathcal{S}}$. Let $\mathbf{T}$ be the K-split torus corresponding to the diagonal matrices in $\mathbf{G}$. Under the above identification $\Gamma = \mathbf{G}(\mathcal{O}_K) = R_{K/F}(\mathbf{G})(\mathcal{O}_F)$ and $T = \mathbf{T}(K_{\mathcal{S}}) = R_{K/F}(\mathbf{T})(F_{\mathcal{S}})$. For every $v \in \mathcal{S}$ we have that $T_v = \mathbf{T}(K_v)$ is the the group of complex diagonal matrices in $G_v(= \mathrm{SL}_2(\mathbb{C}))$. The F-torus $R_{K/F}(\mathbf{T})$ is not split and contains a maximal 1-dimensional F-split torus $\mathbf{T}_F$. Note that $\mathbf{T_F}(F_v)$, $v \in \mathcal{S}$, is the the group of *real* diagonal matrices in T_v. Denote $T_{\mathbb{R}} = \mathbf{T}_F(F_{\mathcal{S}})$. Then $T = T_{\mathbb{R}} \cdot N$ where N is a compact group.

8.2. **Non-homogeneous T-orbits closures when $r > 2$.** Further on, we use the notation and the assumptions from §8.1. Also, $u^-(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$ and $u^+(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ where $x \in K$ or K_v, $\mathcal{S} = \{v_1, \cdots, v_r\}$ and $G = G_{v_1} \times \cdots \times G_{v_r}$. The next theorem is similar to [T3, Theorem 1.8].

Theorem 8.1. *Suppose that $r > 2$. Let $g = (u^-(\beta)u^+(\alpha), e, \cdots, e) \in G$ where $\alpha \in F^*$ and $\beta \in K \setminus F$. Then the following holds:*

(a) *Each of the orbits $\overline{T\pi(g)}$ and $\overline{T_{\mathbb{R}}\pi(g)}$ is not dense in G/Γ,*
(b) *Each of the sets $\overline{T\pi(g)} \setminus T\pi(g)$ and $\overline{T_{\mathbb{R}}\pi(g)} \setminus T_{\mathbb{R}}\pi(g)$ is not contain in a union of countably many closed orbits of proper subgroups of G.*

In particular, each of the closures $\overline{T\pi(g)}$ and $\overline{T_{\mathbb{R}}\pi(g)}$ is not an orbit of a closed subgroup of G.

Proof. A direct calculation shows that $u^-(\beta)u^+(\alpha) = du^+(\alpha_1)u^-(\beta_1)$ where $\alpha_1 = (1+\alpha\beta)\alpha$, $\beta_1 = (1+\alpha\beta)^{-1}\beta$ and $d = \begin{pmatrix} (1+\alpha\beta)^{-1} & 0 \\ 0 & 1+\alpha\beta \end{pmatrix}$. Since $\beta \in K \setminus F$ we get that $\beta_1 \in K \setminus F$.

Define subgroups L_1 and L_2 of G as follows. Put $L_1 = \mathbf{SL}_2(F_{\mathcal{S}})$ and $L_2 = \{\begin{pmatrix} x & y\beta_1^{-1} \\ z\beta_1 & t \end{pmatrix} \in G : x, y, z, t \in F_{\mathcal{S}}\}$. The group $L_1 \cap \Gamma$ is commensurable with $\mathrm{SL}_2(\mathcal{O}_F)$ and, therefore, is a lattice in L_1. This implies that $L_1\pi(e)$ is closed. Since the map $\mathbf{G} \to \mathbf{G}$, $\begin{pmatrix} x & y \\ z & t \end{pmatrix} \mapsto \begin{pmatrix} x & y\beta_1^{-1} \\ z\beta_1 & t \end{pmatrix}$, is a K-isomorphism we get that $L_2\pi(e)$ is closed too.

It follows from the definitions of L_1 and L_2 that

(16)
$$T_{\mathbb{R}}\pi(g) \subset \bigcup_{0\le\mu\le 1}\{(u^-(\mu\beta),\cdots,e)L_1\pi(e)\}\bigcup\bigcup_{0\le\nu\le 1}\{(d\cdot u^+(\nu\alpha_1),\cdots,e)L_2\pi(e)\}.$$

Since the right hand side of (16) is a proper closed subset of G/Γ, $\overline{T_{\mathbb{R}}\pi(g)} \neq G/\Gamma$. Also, it is easy to see that the shift of the right hand side of (16) by the compact group N (defined at the end of §8.1) remains a proper subset of G/Γ. Since $T = T_{\mathbb{R}}\cdot N$, $\overline{T\pi(g)} \neq G/\Gamma$, completing the proof of (a).

Let ${U_1}^+$ be the group consisting of all upper triangular unipotent matrices in L_1 and ${U_2}^-$ be the group consisting of all lower triangular unipotent matrices in L_2. It follows from Proposition 6.10 and Proposition 3.2(2a) that $\overline{T_{\mathbb{R}}^{\circ}\pi(g)} \supset U_1^+\pi(e)\cup(d,e,\cdots,e)U_2^-\pi(e)$. In view of Proposition 6.13 we get that

$$\overline{T_{\mathbb{R}}^{\circ}\pi(g)} \supset L_1\pi(e)\cup(d,e,\cdots,e)L_2\pi(e). \tag{17}$$

Choose a real transcendental number a. Using again Proposition 3.2 we obtain that $\overline{T_{\mathbb{R}}\pi(g)}$ contains $\pi(\widetilde{g})$ where $\widetilde{g} = (u^-(a^{-2}\beta)u^+(a^2\alpha),\cdots,e)$. Remark that

$$\overline{T_{\mathbb{R}}^{\circ}\pi(\widetilde{g})} = \overline{T_{\mathbb{R}}^{\circ}\pi(g)}.$$

Suppose that $\pi(\widetilde{g}) \in T\pi(g)$. Then there exist $t = \begin{pmatrix}\tau & 0\\ 0 & \tau^{-1}\end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$ and $m \in \mathrm{SL}_2(K)$ such that $u^-(\beta)u^+(\alpha)m = tu^-(a^{-2}\beta)u^+(a^2\alpha)$. The upper left coefficient of $tu^-(a^{-2}\beta)u^+(a^2\alpha)$ is equal to τ and $u^-(\beta)u^+(\alpha)m \in \mathrm{SL}_2(K)$. Hence $\tau \in K$. On the other hand, the upper right coefficient of $tu^-(a^{-2}\beta)u^+(a^2\alpha)$ is equal to $\tau a^2\alpha$. Therefore a is an algebraic number which is a contradiction. We have proved that

$$\pi(\widetilde{g}) \in \overline{T_{\mathbb{R}}^{\circ}\pi(g)} \setminus T\pi(g).$$

Since

$$\overline{T_{\mathbb{R}}^{\circ}\pi(g)} \setminus T\pi(g) \subset (\overline{T_{\mathbb{R}}^{\circ}\pi(g)} \setminus T_{\mathbb{R}}^{\circ}\pi(g)) \cap (\overline{T\pi(g)} \setminus T\pi(g)),$$

in order to prove (b) it is enough to show that if $\overline{T_{\mathbb{R}}^{\circ}\pi(g)} \setminus T\pi(g) \subset \bigcup_{i=1}^{\infty} Q_i\pi(h_i)$, where Q_i are connected closed subgroups of G and $Q_i\pi(h_i)$ are closed orbits, then one of the subgroups Q_i is equal to G. It follows from Baire's category theorem that there exists $i_\circ$ such that $T_{\mathbb{R}}^{\circ} \subset Q_{i_\circ}$ and $\pi(\widetilde{g}) \in Q_{i_\circ}\pi(h_{i_\circ})$. Using (17), we obtain that $L_1 \cup (d,e,\cdots,e)L_2(d,e,\cdots,e)^{-1} \subset Q_{i_\circ}$. Since $\beta_1 \in K\setminus F$ using the definitions of L_1 and L_2 we get that $Q_{i_\circ}$ contains $\{e\}\times\cdots\times\{e\}\times \mathbf{G}(K_{v_r})$. (Recall that $\mathbf{G}(K_{v_r}) = \mathrm{SL}_2(\mathbb{C})$.) Since Γ is an irreducible lattice of G, $(\{e\}\times\cdots\times\{e\}\times\mathbf{G}(K_r))\cdot\Gamma$ is dense in G. Therefore $Q_{i_\circ} = G$, completing the proof of the theorem. $\square$

Remark. The orbit $T\pi(g)$, considered as an orbit on $\mathbf{SL}_2(K_{\mathcal{S}})/\mathbf{SL}_2(\mathcal{O}_K)$, provides an example showing that Corollary 1.9 is not valid for CM-fields. On the

other hand, $T_{\mathbb{R}}\pi(g)$, considered as an orbit on $R_{K/F}(\mathbf{SL_2})(F_{\mathcal{S}})/R_{K/F}(\mathbf{SL_2})(\mathcal{O}_F)$, shows that $\overline{T\pi(g)}$ in the formulation of Theorem 1.8 is not always homogeneous.

8.3. **Values of decomposable forms when $\#\mathcal{S} = 2$ or $\#\mathcal{S} \geq 2$ and K is a CM-field.** Let us provide the necessary counter-examples showing that the assertion of Theorem 1.10 does not hold if $\#\mathcal{S} = 2$ or K is a CM-field and $\#\mathcal{S} \geq 2$.

We keep the notation $f(\vec{x})$, $f_v(\vec{x})$ and $l_i^{(v)}(\vec{x})$ as in the formulation of Theorem 1.10. For simplicity, we assume that $m = n = 2$.

The following is a particular case of [T3, Theorem 1.10]:

Theorem 8.2. *Let $\#\mathcal{S} = 2$. Then $\overline{f(\mathcal{O}_K^2)} \cap K_{\mathcal{S}}^*$ is a countable set. In particular, $\overline{f(\mathcal{O}_K^2)}$ is not dense in $K_{\mathcal{S}}^*$.*

Remark that in the formulation of Theorem 8.2 K is not necessarily a CM-field.

Theorem 8.3. *Let K be a* CM*-field which is a quadratic extension of a totally real field F. We suppose that $\#\mathcal{S} \geq 2$ and all $l_i^{(v)}(\vec{x})$ are with coefficients from F. There exists a real $C > 0$ such that if $\vec{z} \in \mathcal{O}_K^2$ and $f(\vec{z}) \in K_{\mathcal{S}}^*$ then either*

$$\prod_{v \in \mathcal{S}} |f_v(\vec{z})|_v \geq C \tag{18}$$

or there exists $w \in \mathbb{C}$ such that

$$f_v(\vec{z}) \in \mathbb{R}w \text{ for all } v \in \mathcal{S}. \tag{19}$$

In particular, $\overline{f(\mathcal{O}^2)}$ is not dense in $K_{\mathcal{S}}$.

Proof. Choose $d \in F$ such that $K = F(\sqrt{-d})$ and $\sqrt{-d} \in \mathcal{O}_K$. Put $l = [\mathcal{O}_K : \mathcal{O}_F(\sqrt{-d})]$. If $\mathcal{S} = \{v_1, \cdots, v_r\}$ we use the simpler notation: $K_j := K_{v_j}$, $|\cdot|_j := |\cdot|_{v_j}$, $f_j := f_{v_j}$ and $l_i^{(j)} := l_i^{(v_j)}$. Let $l_i^{(j)}(x_1, x_2) = h_{i1}^{(j)} x_1 + h_{i2}^{(j)} x_2$ where $j \in \{1, \cdots, r\}$ and $i \in \{1, 2\}$. Put $h^{(j)} := \begin{pmatrix} h_{11}^{(j)} & h_{12}^{(j)} \\ h_{21}^{(j)} & h_{22}^{(j)} \end{pmatrix}$. After multiplying f_j by appropriate elements from F^* we may (and will) suppose that $h^{(j)} \in \mathrm{SL}_2(F)$ for all j. Further on, if $\vec{w}_1 = (w_{11}, w_{12}) \in \mathbb{C}^2$ and $\vec{w}_2 = (w_{21}, w_{22}) \in \mathbb{C}^2$ we denote by $\det(\vec{w}_1, \vec{w}_2)$ the determinant of $\begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}$. Also, given $w = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}$ and $\vec{a} = (a_1, a_2) \in \mathbb{C}^2$ we write $w(\vec{a}) = (w_{11}a_1 + w_{12}a_2, w_{21}a_1 + w_{22}a_2)$.

With $C = \left(\frac{d}{4l^4}\right)^r$ we will prove that given $\vec{z} \in \mathcal{O}_K^2$ one of (18) or (19) holds. Let $\vec{z} = \vec{\gamma} + \sqrt{-d}\vec{\delta} \in \mathcal{O}_K^2$ where $\vec{\gamma} = (\gamma_1, \gamma_2) \in F^2$ and $\vec{\delta} = (\delta_1, \delta_2) \in F^2$. By the choice of l, $(l\gamma_1, l\gamma_2) \in \mathcal{O}_F^2$ and $(l\delta_1, l\delta_2) \in \mathcal{O}_F^2$. It is well known that if $\alpha \in \mathcal{O}_F$

then $\prod_j |\alpha|_j \in \mathbb{N}$ (cf.[CF, ch.2, Theorem 11.1]). Therefore

$$\prod_j |\det(h^{(j)}(\vec{\gamma}), h^{(j)}(\vec{\delta}))|_j = \prod_j |\det(\vec{\gamma}, \vec{\delta})|_j \in \frac{1}{l^{4r}}\mathbb{N}.$$

We have

$$h^{(j)}(\vec{z}) = h^{(j)}(\vec{\gamma}) + \sqrt{-d}h^{(j)}(\vec{\delta}) = (r_1^{(j)}e^{\mathrm{i}\varphi_1^{(j)}}, r_2^{(j)}e^{\mathrm{i}\varphi_2^{(j)}}) \in \mathbb{C}^2,$$

where $r_1^{(j)}$ and $r_2^{(j)}$ are the absolute values and $\varphi_1^{(j)}$ and $\varphi_2^{(j)}$ are the arguments of the complex coordinates of $h^{(j)}(\vec{z})$. Since $f_j(\vec{z}) = r_1^{(j)} r_2^{(j)} e^{\mathrm{i}(\varphi_1^{(j)}+\varphi_2^{(j)})}$, we get

$$\det(h^{(j)}(\vec{\gamma}), h^{(j)}(\vec{\delta})) = \frac{f_j(\vec{z})}{\sqrt{d}} e^{-\mathrm{i}(\varphi_1^{(j)}+\varphi_2^{(j)})} \det \begin{pmatrix} \cos\varphi_1^{(j)} & \sin\varphi_1^{(j)} \\ \cos\varphi_2^{(j)} & \sin\varphi_2^{(j)} \end{pmatrix}.$$

Therefore

$$\prod_j |f_j(\vec{z})|_j \geq \left(\frac{d}{4}\right)^r \prod_j |\det(\vec{\gamma}, \vec{\delta})|_j \in \left(\frac{d}{4l^4}\right)^r \mathbb{N}.$$

So, (18) holds unless $\det(\vec{\gamma}, \vec{\delta}) = 0$. In the latter case $\vec{\gamma}$ and $\vec{\delta}$ are proportional which implies (19). □

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